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ABSTRACT

The only way to see what is different about the new ways of doing mathematics is to study some new mathematics presented in a new way. This small text is designed for parents to enable them to study some of the new mathematics topics. The two chapters which are included are extracted from a seventh grade SMSG text. Interspersed throughout are remarks concerned with both the why and the how of the student material. Chapter 1 is on numeration. Chapter 2 is concerned with whole numbers. (Author/RH)

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SCHOOL MATHEMATICS STUDY GROUP

VERY SHORT COURSE IN MATHEMATICS FOR PARENTS

Edited by
E. BEGLE

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Edited by E. BEGLE

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PREFACE

Each fall a large number of parents discover that something has happened to the mathematics courses their children are studying in school. Homework assignments contain new words and ideas, and parents often find that they can no longer help their children do their arithmetic problems.

Such a situation naturally raises a host of questions in the minds of parents. Most of these have already been answered in print (see "The Revolution in School Mathematics," National Council of Teachers of Mathematics, 1201 - 16th Street, N. W., Washington 6, D. C.), but there are some which defy brief explanation. Two of them are: "What are some of the new ideas, and what good are they?" and "What's different about the new way of doing mathematics?" The purpose of this booklet is to provide parents with some answers to these questions.

The only way to see what is different about the new way of doing mathematics is to study some mathematics presented in a new way. The only way to understand what one of the new topics really is is to study it. This booklet is therefore a miniature textbook for parents and should be used as such.

The two chapters which follow have been extracted from a seventh grade text prepared by the School Mathematics Study Group. The first is devoted to a topic which is contained in many of the new mathematics programs but which was not, until recently, a normal part of the curriculum. The second chapter is an illustration of a new way of treating an old topic, the arithmetic of whole numbers.

Interspersed throughout these chapters are some remarks, which are indicated, as here, by a heavy dark line in the margin. These remarks, originally addressed to the teacher, are concerned with both the why and the how of the student material.

The nature of mathematics is such that one learns very little by a casual reading of the pages. It is necessary to study the text carefully, read slowly, perhaps take notes, and above all to work the problems. (Answers are at the end of the booklet.) Hundreds of thousands of seventh graders have already studied this material with profit and pleasure. We hope you will enjoy it also.

Chapter 1

NUMERATION

For this unit little background is needed except familiarity with the number symbols and the basic operations with numbers. The purpose of the unit is to deepen the pupil's understanding of the decimal notation for whole numbers, especially with regard to place value, and thus to help him delve a little deeper into the reasons for the procedures, which he already knows, for carrying out the addition and multiplication operations. One of the best ways to accomplish this is to consider systems of number notations using bases other than ten. Since, in using a new base, the pupil must necessarily look at the reasons for "carrying" and the other mechanical procedures in a new light, he should gain deeper insight into the decimal system. A certain amount of computation in other systems is necessary to "fix" these ideas, but such computation should not be regarded as an end in itself.

The decimal system is used in most of the world today because it is a better system than any other number system we know of. Therefore, it is important that you understand the system and know how to read and write numerals in this system.

Long ago man learned that it was easier to count large numbers of objects by grouping the objects. We use the same idea today when we use a dime to represent a group of ten pennies, and a dollar to represent a group of ten dimes. Because we have ten fingers it is natural for us to count by tens. We use ten symbols for our numerals. These symbols, which are called digits, are 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0. The word digit refers to our fingers and to these ten number symbols. With these ten symbols we can write a number as large or as small as we wish.

The decimal system uses the idea of place value to represent the size of a group. The size of the group represented by a symbol depends upon the position of the symbol or digit in a numeral. The symbol tells us how many of that group we have. In the numeral 123, the "1" represents one group of one hundred; the "2" represents two groups of ten, or twenty; and the "3" represents three ones, or three.

Since we group by tens in the decimal system, we say its base is ten. Because of this, each successive (or next) place to the left represents a group ten times that of the preceding place. The first place tells us how many groups of one. The second place tells us how many groups of ten, or ten times one (10×1). The third place tells us how many groups of ten times ten (10×10), or one hundred, the next, ten times ten times ten ($10 \times 10 \times 10$), or one thousand, and so on. By using a base and the ideas of place value, it is possible to write any number in the decimal system using only the ten basic symbols. There is no limit to the size of numbers which can be represented by the decimal system.

To understand the meaning of the number represented by a numeral such as 123 we add the numbers represented by each symbol. Thus 123 means $(1 \times 100) + (2 \times 10) + (3 \times 1)$, or $100 + 20 + 3$. The same number is represented by $100 + 20 + 3$ and by 123. When we write a numeral such as 123 we are using number symbols, the idea of place value, and base ten.

One advantage of our decimal system is that it has a symbol for zero. Zero is used to fill places which would otherwise be empty and might lead to misunderstanding. In writing the numeral for three hundred seven, we write 307. Without a symbol for zero we might find it necessary to write 3-7. The meaning of 3-7 or 3 7 might be confused. The origin of the idea of zero is uncertain, but the Hindus were using a symbol for zero about 600 A.D., or possibly earlier.

The clever use of place value and the symbol for zero makes the decimal system one of the most efficient systems in the world. Pierre Simon Laplace (1749 - 1827), a famous French mathematician, called the decimal system one of the world's most useful inventions.

1-2. Expanded Numerals and Exponential Notation.

Exponents are introduced here in a situation which shows clearly their usefulness for concise notation. Furthermore, their use serves to emphasize the role of the base and of position. This role will be more fully utilized in the sections to follow.

We say that the decimal system of writing numerals has a base ten. Starting at the units place, each place to the left has a value ten times as large as the place to its right. The first six places from the right to the left are shown below:

hundred thousand	ten thousand	thousand	hundred	ten	one
(10×10×10×10×10)	(10×10×10×10)	(10×10×10)	(10×10)	(10)	(1)

Often we write these values more briefly, by using a small numeral to the right and above the 10. This numeral shows how many 10's are multiplied together. Numbers that are multiplied together are called factors. In this way, the values of the places are written and read as follows:

$(10 \times 10 \times 10 \times 10 \times 10)$	10^5	"ten to the fifth power"
$(10 \times 10 \times 10 \times 10)$	10^4	"ten to the fourth power"
$(10 \times 10 \times 10)$	10^3	"ten to the third power"
(10×10)	10^2	"ten to the second power"
(10)	10^1	"ten to the first power"
(1)	1	"one"

In an expression as 10^2, the number 10 is called the base and the number 2 is called the exponent. The exponent tells how many times the base is taken as a factor in a product. 10^2 indicates (10×10) or 100. A number such as 10^2 is called a power of ten, and in this case it is the second power of ten. The exponent is sometimes omitted for the first power of ten; we usually write 10, instead of 10^1. All other exponents are always written. Another way to write $(4 \times 4 \times 4)$ is 4^3, where 4 is the base, and 3 is the exponent.

How can we write the meaning of "352" with exponents?

$$352 = (3 \times 10 \times 10) + (5 \times 10) + (2 \times 1)$$
$$= (3 \times 10^2) + (5 \times 10^1) + (2 \times 1).$$

This is called expanded notation. Writing numerals in expanded notation helps explain the meaning of the whole numeral.

1-3. Numerals in Base Seven.

The purpose of teaching systems of numeration in bases other than ten is not to produce facility in calculating with such systems. A study of an unfamiliar system aids in understanding a familiar one, just as the study of a foreign language aids us in understanding our own. The decimal system is so familiar that its structure and the ideas involved in its algorithms are easily overlooked. In this section attention is focused on numerals, rather than on numbers.

You have known and used decimal numerals for a long time, and you may think you understand all about them. Some of their characteristics, however, may have escaped your notice simply because the numerals are familiar to you. In this section you will study a system of notation with a different base. This will increase your understanding of decimal numerals.

Suppose we found people living on Mars with seven fingers. Instead of counting by tens, a Martian might count by sevens. Let us see how to write numerals in base seven notation. This time we plan to work with groups of seven. Look at the x's below and notice how they are grouped in sevens with some x's left over.

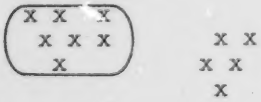


Figure 1-3a

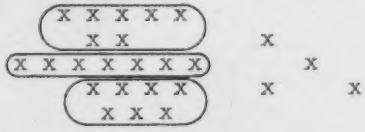


Figure 1-3b

In Figure 1-3a, we see one group of seven and five more. The numeral is written 15_{seven}. In this numeral, the 1 shows that there is one group of seven, and the 5 means that there are five ones.

In Figure 1-3b, how many groups of seven are there? How many x's are left outside the groups of seven? The numeral representing this number of x's is 34_{seven}. The 3 stands for three groups of seven, and the 4 represents four single x's or four ones. The "lowered" seven merely shows that we are working in base seven.

When we group in sevens the number of individual objects left can only be zero, one, two, three, four, five, or six. Symbols are needed to represent those numbers. Suppose we use the familiar 0, 1, 2, 3, 4, 5 and 6 for these, rather than invent new symbols. As you will discover, no other symbols are needed for the base seven system.

If the x's are marks for days, we may think of 15_{seven} as a way of writing 1 week and five days. In our decimal system we name this number of days "twelve" and write it "12" to show one group of ten and two more. We do not write the base name in our numerals since we all know what the base is.

We should not use the name "fifteen" for 15_{seven} because fifteen is 1 ten and 5 more. We shall simply read 15_{seven} as "one, five, base seven."

You know how to count in base ten and how to write the numerals in succession. Notice that one, two, three, four, five, six, seven, eight, and nine are represented by single symbols. How is the base number "ten" represented? This representation, 10, means one group of ten and zero more.

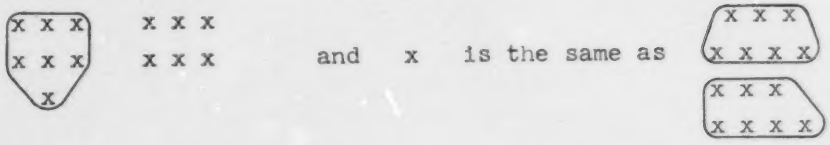
With this idea in mind, think about counting in base seven. Try it yourself and compare with the following table, filling in the numerals from 21_{seven} to 63_{seven}. In this table the "lowered" seven is omitted.

Counting in Base Seven

Number	Symbol	Number	Symbol
one	1	one, four	14
two	2	one, five	15
three	3	one, six	16
four	4	two, zero	20
five	5	two, one	21
six	6	---	--
one, zero	10	six, three	63
one, one	11	six, four	64
one, two	12	six, five	65
one, three	13	six, six	66

How did you get the numeral following 16_{seven}?

You probably thought something like this:



which is 2 groups of seven x's and 0 x's left over.

What would the next numeral after 66_{seven} be? Here you would have 6 sevens and 6 ones plus another one. This equals 6 sevens and another seven, that is, seven sevens. How could we represent $(seven)^2$ without using a new symbol? We introduce a new group, the $(seven)^2$ group. This number would then be written 100_{seven}. What does the number really mean? Go on from this point and write a few more numbers. What would be the next numeral after 666_{seven}?

Place Values in Base Seven

$(seven)^5$	$(seven)^4$	$(seven)^3$	$(seven)^2$	$(seven)^1$	(one)

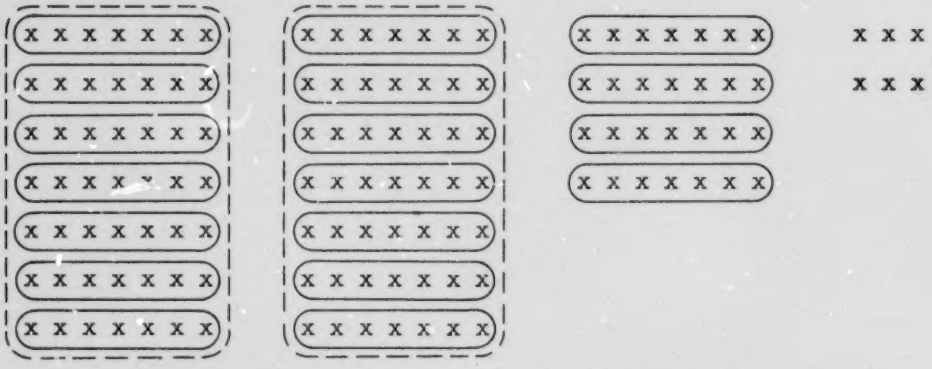
Notice that each place represents seven times the value of the next place to the right. The first place on the right is the <u>one place</u> in both the decimal and the seven systems. The value of the second place is the <u>base times one</u>. In this case what is it? The value in the third place from the right is (seven × seven), and in the next place (seven × seven × seven).

What is the decimal name for (seven × seven)? We need to use this (forty-nine) when we change from base seven to base ten. Show that the decimal numeral for $(seven)^3$ is 343. What is the decimal numeral for $(seven)^4$?

Using the chart above, we see that

$$246_{seven} = (2 \times seven \times seven) + (4 \times seven) + (6 \times one).$$

The diagram shows the actual grouping represented by the digits and the place values in the numeral 246_{seven}:



(2 × seven × seven) (4 × seven) (6 × one)

If we wish to write the number of x's above in the decimal system of notation we may write:

$$\begin{aligned} 246_{seven} &= (2 \times 7 \times 7) + (4 \times 7) + (6 \times 1) \\ &= (2 \times 49) + (4 \times 7) + (6 \times 1) \\ &= 98 + 28 + 6 \\ &= 132_{ten}. \end{aligned}$$

Regroup the x's above to show that there are 1 (ten × ten) group, 3(ten) groups, and 2 more. This should help you understand that $246_{seven} = 132_{ten}$.

Exercises 1-3

1. Group the x's below and write the number of x's in base seven notation:

a. x x x x x
x x x x x

b. x x x x x
x x x x x
x x x x
x x x x

c. x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x

2. Draw x's and group them to show the meaning of the following numerals.

a. 11_{seven} b. 26_{seven} c. 35_{seven} d. 101_{seven}.

3. Write each of the following numerals in expanded form and then in decimal notation.

   a. 33_{seven} b. 45_{seven} c. 100_{seven} d. 524_{seven}.

4. Write the next consecutive numeral after each of the following numerals.

   a. 6_{seven} c. 54_{seven} e. 666_{seven}

   b. 10_{seven} d. 162_{seven} f. 1006_{seven}.

5. What is the value of the "6" in each of the following numerals?

   a. 560_{seven} b. 56_{seven} c. 605_{seven} d. 6050_{seven}.

6. In the base seven system write the value of the fifth place counting left from the units place.

7. In the base seven system, what is the value of the tenth place from the right?

*8. What numeral in the seven system represents the number named by six dozen?

9. Which number is larger, 452_{seven} or 432_{seven}?

10. Which number is greater, 250_{seven} or 205_{ten}?

11. Which is smaller, 2125_{seven} or 754_{ten}?

12. BRAINBUSTER. On planet X-101 the pages in books are numbered in order as follows: |, ∠, △, □, ⧅, ⊠, |-, ||, |∠, |△, |□, |⧅, |⊠, ∠-, ∠|, and so forth. What seems to be the base of the numeration system these people use? Why? How would the next number after ∠| be written? Which symbol corresponds to our zero? Write numerals for numbers from □- to ⧅△.

1-4. Computation in Base Seven.

Computation in base seven is undertaken to clarify computation in decimal notation. The stress here should be on the ideas involved and not on computational proficiency.

Addition

In the decimal, or base ten, system there are 100 "basic" addition combinations. By this time you know all of them. The combinations can be arranged in a convenient table. Part of the table is given below.

Addition, Base Ten

+	0	1	2	3	4	5	6	7	8	9
0	0									
1	1	2								
2				5						
3	3	4	5	6						
4	4	5	6	7	8					
5	5	6	7	8	9	10	11			
6										
7										
8										
9									17	

The numbers represented in the horizontal row above the line at the top of the table are added to the numbers in the vertical row under the "+" sign at the left. The sum of each pair of numbers is written in the table. The sum 2 + 3 is 5, as pointed out by the arrows.

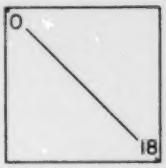
Exercises 1-4a

1. Find the sums

   a. 6 + 5 b. 9 + 8.

2. Use cross ruled paper and complete the addition table on page 9 (you will use it later).

3. Draw a diagonal line from the upper left corner to the lower right corner of the chart as shown at the right.



   a. Is $3 + 4$ the same as $4 + 3$?
   b. How could the answer to part a) be determined from the chart?
   c. What do you notice about the two parts of the chart?
   d. What does this tell you about the number of <u>different</u> combinations which must be mastered? Be sure you can recall any of these combinations whenever you need them.

4. Make a chart to show the basic sums when the numbers are written in base seven notation. Four sums are supplied to help you.

+	0	1	2	3	4	5	6
0							
1				4			
2					6		
3						11	
4							13
5							
6							

5. a. How many <u>different</u> number combinations are there in the base seven table? Why?
   b. Which would be easier, to learn the necessary multiplication combinations in base seven or in base ten? Why?
   c. Find $4_{ten} + 5_{ten}$ and $4_{seven} + 5_{seven}$ from the tables. Are the results equal? That is, do they represent the same number?

The answer to problem 5c is an illustration of the fact that a number is an idea independent of the numerals used to write its name. Actually, 9_{ten} and 12_{seven} are two different names for the same number.

Do not try to memorize the addition combinations for base seven. The value in making the table lies in the help it gives you in understanding operations with numbers.

The table that you completed in problem 4 of the last set of exercises shows the sums of pairs of numbers from zero to six. Actually, little more is needed to enable us to add larger numbers. In order to see what else is needed, let us consider how we add in base ten. What are the steps in your thinking when you add numbers like twenty-five and forty-eight in the decimal notation?

$$\begin{array}{rcl} 25 & = & 2 \text{ tens} + 5 \text{ ones} \\ \underline{48} & = & \underline{4 \text{ tens} + 8 \text{ ones}} \\ & & 6 \text{ tens} + 13 \text{ ones} = 7 \text{ tens} + 3 \text{ ones} \end{array} \quad \begin{array}{l} = \longrightarrow 25 \\ = \longrightarrow \underline{48} \\ = 73 \end{array}$$

Try adding in base seven: $14_{\text{seven}} + 35_{\text{seven}}$

$$\begin{array}{l} 1 \text{ seven} + 4 \text{ ones} \\ \underline{3 \text{ sevens} + 5 \text{ ones}} \\ 4 \text{ sevens} + 12 \text{ ones} = 5 \text{ sevens} + 2 \text{ ones} = 52_{\text{seven}}. \end{array}$$

(You may look up the sums $5 + 4$ and $3 + 1$ in the base seven addition table.)

How are the two examples alike? How are they different? When is it necessary to "carry" (or regroup) in the ten system? When is it necessary to "carry" (or regroup) in the seven system?

Try your skill in addition on the following problems. Use the addition table for the basic sums.

$$\begin{array}{lllll} 42_{\text{seven}} & 65_{\text{seven}} & 32_{\text{seven}} & 254_{\text{seven}} & 435_{\text{seven}} \\ \underline{13}_{\text{seven}} & \underline{11}_{\text{seven}} & \underline{25}_{\text{seven}} & \underline{105}_{\text{seven}} & \underline{625}_{\text{seven}} \end{array} \qquad \begin{array}{l} 524_{\text{seven}} \\ \underline{564}_{\text{seven}} \end{array}$$

The answers in order are 55_{seven}, 106_{seven}, 60_{seven}, 362_{seven}, 1363_{seven}, and 1421_{seven}.

Subtraction

How did you learn to subtract in base ten? You probably used subtraction combinations such as $14 - 5$ until you were thoroughly familiar with them. You know the answer to this problem but suppose, for the moment, that you did not. Could you get the answer from the addition table? You really want to ask the following question: "What is the number which, when added to 5, yields 14?" Since the seventh row of the base ten addition

table gives the results of adding various numbers to 5, we should look for 14 in that row. Where do you find the answer to $14 - 5$? Did you answer "the last column"? Use the base ten addition table to find

$9 - 2$, $8 - 5$, $12 - 7$, $17 - 9$.

The idea discussed above is used in every subtraction problem. One other idea is needed in many problems, the idea of "borrowing" or "regrouping." This last idea is illustrated below for base ten to find $761 - 283$:

$$\begin{aligned} 7 \text{ hundreds} + 6 \text{ tens} + 1 \text{ one} &= 6 \text{ hundreds} + 15 \text{ tens} + 11 \text{ ones} = 761 \\ \underline{2 \text{ hundreds} + 8 \text{ tens} + 3 \text{ ones}} &= \underline{2 \text{ hundreds} + 8 \text{ tens} + 3 \text{ ones}} = \underline{283} \\ & 4 \text{ hundreds} + 7 \text{ tens} + 8 \text{ ones} = 478 \end{aligned}$$

Now let us try subtraction in base seven. How would you find $6_{\text{seven}} - 2_{\text{seven}}$? Find $13_{\text{seven}} - 6_{\text{seven}}$. How did you use the addition table for base seven? Find answers to the following subtraction examples:

$$\begin{array}{lllll} 15_{\text{seven}} & 12_{\text{seven}} & 11_{\text{seven}} & 14_{\text{seven}} & 13_{\text{seven}} \\ \underline{\quad 6_{\text{seven}}} & \underline{\quad 4_{\text{seven}}} & \underline{\quad 6_{\text{seven}}} & \underline{\quad 5_{\text{seven}}} & \underline{\quad 4_{\text{seven}}} \end{array}$$

The answers to these problems are 6_{seven}, 5_{seven}, 2_{seven}, 6_{seven}, and 6_{seven}.

Let us work a harder subtraction problem in base seven comparing the procedure with that used above:

$$\begin{aligned} 43_{\text{seven}} &= 4 \text{ sevens} + 3 \text{ ones} = 3 \text{ sevens} + 13 \text{ ones} = 43_{\text{seven}} \\ \underline{16_{\text{seven}}} &= \underline{1 \text{ seven} + 6 \text{ ones}} = \underline{1 \text{ seven} + 6 \text{ ones}} = \underline{16_{\text{seven}}} \\ & \qquad\qquad 2 \text{ sevens} + 4 \text{ ones} = 24_{\text{seven}} \end{aligned}$$

Be sure to note that "13 ones" above is in the seven system and is "one seven, three ones." If you wish to find the number you add to 6_{seven} to get 13_{seven}, how can you use the table to help you? Some of you may think of the number without referring to the table.

Practice on these subtraction examples:

56_{seven}	61_{seven}	34_{seven}	452_{seven}	503_{seven}
$\underline{14}_{seven}$	$\underline{35}_{seven}$	$\underline{26}_{seven}$	$\underline{263}_{seven}$	$\underline{140}_{seven}$

The answers are 42_{seven}, 23_{seven}, 5_{seven}, 156_{seven} and 333_{seven}.

Exercises 1-4b

1. Each of the following examples is written in base seven. Add. Check by changing the numerals to decimal notation and adding in base ten as in the example:

Base Seven		Base Ten
16_{seven}	=	13
$\underline{23}_{seven}$	=	$\underline{17}$
42_{seven}		30

Does $42_{seven} = 30$?

a. 25_{seven} b. 56_{seven} c. 214_{seven}
$\underline{31}_{seven}$ $\underline{21}_{seven}$ $\underline{53}_{seven}$

d. $160_{seven} + 430_{seven}$ e. $45_{seven} + 163_{seven}$

f. $403_{seven} + 563_{seven}$ g. $645_{seven} + 605_{seven}$

h. $6245_{seven} + 5314_{seven}$ i. $6204_{seven} + 234_{seven}$

j. $645_{seven} + 666_{seven}$ k. $5406_{seven} + 6245_{seven}$.

2. Use the base seven addition table to find:

a. $6_{seven} - 4_{seven}$ b. $11_{seven} - 4_{seven}$

c. $12_{seven} - 5_{seven}$.

3. Each of the following examples is written in base seven. Subtract. Check by changing to decimal numerals.

a. 10_{seven} $-\ 5_{seven}$

b. 65_{seven} $-\ 26_{seven}$

c. 200_{seven} $-\ 4_{seven}$

d. 160_{seven} $-\ 6_{seven}$

e. $44_{seven} - 35_{seven}$

f. $641_{seven} - 132_{seven}$

g. $502_{seven} - 266_{seven}$

h. $5000_{seven} - 4261_{seven}$

i. $634_{seven} - 52_{seven}$

j. $134_{seven} - 65_{seven}$

k. $3451_{seven} - 2164_{seven}$

ℓ. $253_{seven} - 166_{seven}$

4. Show by grouping x's that:

a. 4 twos = 11_{seven}

b. 6 threes = 24_{seven}

c. 3 fives = 21_{seven}

d. 5 sixes = 42_{seven}

Multiplication

In order to multiply, we may use a table of basic facts. Complete the following table in decimal numerals and be sure you know and can recall instantly the product of any two numbers from zero to nine.

Multiplication, Base Ten

x	0	1	2	3	4	5	6	7	8	9
0	0	0								
1	0	1								
2			4	6						
3				9	12					
4										
5										
6										
7										
8										
9										

Exercises 1-4c

1. Refer to the preceding table.
   a. Explain the row of zeros and the column of zeros.
   b. Which row in the table is exactly like the row at the top? Why?
2. Imagine a diagonal line drawn from the × sign in the table to the lower right corner. What can you say about the two triangular parts of the table on each side of the line?
3. Complete the multiplication table below for base seven. Suggestion: To find $4_{seven} \times 3_{seven}$ you could write four x's three times and regroup to show the base seven numeral. Better still, you might think of this as $3_{seven} + 3_{seven} + 3_{seven} + 3_{seven}$.

Multiplication, Base Seven

×	0	1	2	3	4	5	6
0							
1							
2					11	13	
3							
4				15			
5							
6							51

You know about carrying (or regrouping) in addition, and you have had experience in multiplication in base ten. Use the base seven multiplication table to find the following products.

52_{seven}	34_{seven}	421_{seven}	621_{seven}	604_{seven}
$\times\ 3_{seven}$	$\times\ 6_{seven}$	$\times\ 4_{seven}$	$\times\ 2_{seven}$	$\times\ 35_{seven}$

The answers are 216_{seven}, 303_{seven}, 2314_{seven}, 1542_{seven}, 31406_{seven}.

Check the multiplication shown at the right and then answer the following questions. How do you get the entry 123 on the third line? How do you get the entry 201 on the fourth line? Why is the 1 on line 4 placed under the 2 on line 3? Why is the 0 on line 4 placed under the 1 on line 3? If you do not know

$$\begin{array}{r} 45_{seven} \\ \times\ 32_{seven} \\ \hline 123 \\ 201 \\ \hline 2133_{seven} \end{array}$$

why the entries on lines 3 and 4 are added to get the answer, you will study this more thoroughly later.

Division

Division is left as an exercise for you. You may find that it is not easy. Working in base seven should help you understand why some boys and girls have trouble with division in base ten. Here are two examples you may wish to examine. All the numerals within the examples are written in base seven. How can you use the multiplication table here?

Division in Base Seven

$$6_{seven}\overline{)4053_{seven}}\quad 454_{seven}$$

33
45
42
33
33

$$46_{seven}\overline{)126125_{seven}}\quad 2015_{seven}$$

125
112
46
335
332
3

Exercises 1-4d

1. Multiply the following numbers in base seven numerals and check your results in base ten.

   a. $14_{seven} \times 3_{seven}$
   b. $6_{seven} \times 25_{seven}$
   c. $63_{seven} \times 12_{seven}$
   d. $5_{seven} \times 461_{seven}$
   e. $56_{seven} \times 43_{seven}$
   f. $654_{seven} \times 453_{seven}$
   g. $3046_{seven} \times 24_{seven}$
   h. $5643_{seven} \times 652_{seven}$
   i. $250_{seven} \times 341_{seven}$
   j. $26403_{seven} \times 45_{seven}$

2. Divide. All numerals in this exercise are in base seven.

   a. $6_{seven}\overline{)42_{seven}}$
   b. $5_{seven}\overline{)433_{seven}}$
   c. $4_{seven}\overline{)2316_{seven}}$
   d. $21_{seven}\overline{)2625_{seven}}$

1-5. Changing from Base Ten to Base Seven.

In general, it is easier to change from base seven numerals to base ten numerals than the reverse. Again, the stress should be on the ideas rather than on computational facility.

You have learned how to change a number written in base seven numerals to base ten numerals. It is also easy to change from base ten to base seven. Let us see how this is done.

In base seven, the values of the places are: one, seven^1, seven^2, seven^3, and so on. That is, the place values are one and the powers of seven.

$$\text{seven}^1 = 7_{\text{ten}}$$
$$\text{seven}^2 = (7 \times 7) \text{ or } 49_{\text{ten}}$$
$$\text{seven}^3 = (7 \times 7 \times 7) \text{ or } 343_{\text{ten}}$$

Suppose you wished to change 12_{ten} to base seven numerals. This time we shall think of groups of powers of seven instead of actually grouping marks. What is the largest power of seven which is contained in 12_{ten}? Is seven^1 the largest? How about seven^2 (forty-nine) or seven^3 (three hundred forty-three)? We can see that only seven^1 is small enough to be contained in 12_{ten}.

When we divide 12 by 7 we have

$$\begin{array}{r} 1 \\ 7\overline{)12} \\ \underline{7} \\ 5 \end{array}$$

What does the 1 on top mean? What does the 5 mean? They tell us that 12_{ten} contains 1 seven with 5 units left over, or that $12_{\text{ten}} = (1 \times \text{seven}) + (5 \times \text{one})$. Thus $12_{\text{ten}} = 15_{\text{seven}}$.

Be sure you know which place in a base seven numeral has the value seven^2, the value seven^3, the value seven^4, and so on.

How is 54_{ten} regrouped for base seven numerals? What is the largest power of seven which is contained in 54_{ten}?

In 54_{ten} we have ___?___ × $seven^2$ + ___?___ × seven + ___?___ × one.

$$\begin{array}{r} 1 \\ 49\overline{)54} \\ \underline{49} \\ 5 \end{array}$$

We have (___1___ × $seven^2$) + (0 × seven) + (___5___ × one).
Then $54_{ten} = 105_{seven}$.

Suppose the problem is to change 524_{ten} to base seven numerals. Since 524_{ten} is larger than 343 ($seven^3$), find how many 343's there are.

$$\begin{array}{r} 1 \\ 343\overline{)524} \\ \underline{343} \\ 181 \end{array}$$

Thus 524 contains one $seven^3$ with 181 remaining, or $524 = (1 \times seven^3) + 181$, and there will be a "1" in the $seven^3$ place.

Now find how many 49's ($seven^2$) there are in the remaining 181.

$$\begin{array}{r} 3 \\ 49\overline{)181} \\ \underline{147} \\ 34 \end{array}$$

Thus 181 contains 3 49's with 34 remaining, or $181 = (3 \times seven^2) + 34$, and there will be a "3" in the $seven^2$ place.

How many sevens are there in the remaining 34?

$$\begin{array}{r} 4 \\ 7\overline{)34} \\ \underline{28} \\ 6 \end{array}$$

Thus 34 contains 4 seven's with 6 remaining, or $34 = (4 \times seven) + 6$, and there will be a "4" in the sevens place.

What will be in the units place? We have:

$524_{ten} = (1 \times seven^3) + (3 \times seven^2) + (4 \times seven) + (6 \times one)$

$524_{ten} = 1346_{seven}$.

Cover the answers below until you have made the changes for yourself.

$10_{ten} = (1 \times seven) + (3 \times one) = 13_{seven}$

$46_{ten} = (6 \times seven) + (4 \times one) = 64_{seven}$

$162_{ten} = (3 \times seven^2) + (2 \times seven) + (1 \times one) = 321_{seven}$

$1738_{ten} = (5 \times seven^3) + (0 \times seven^2) + (3 \times seven) + (2 \times one)$

$= 5032_{seven}$.

In changing base ten numerals to base seven we first select the largest place value of base seven (that is, power of seven) contained in the number. We divide the number by this power of seven and find the quotient and remainder. The quotient is the first digit in the base seven numeral. We divide the remainder by the next smaller power of seven and this quotient is the second digit. We continue to divide remainders by each succeeding, smaller power of seven to determine all the remaining digits in the base seven numeral.

Exercises 1-5

1. Show that:

   a. $50_{ten} = 101_{seven}$ b. $145_{ten} = 265_{seven}$

   c. $1024_{ten} = 2662_{seven}$

2. Change the following base ten numerals to base seven numerals:

   a. 12 d. 53
   b. 36 e. 218
   c. 44 f. 1320

1-6. Numerals in Other Bases.

The base of the system we use is "ten" for historical rather than mathematical reasons.

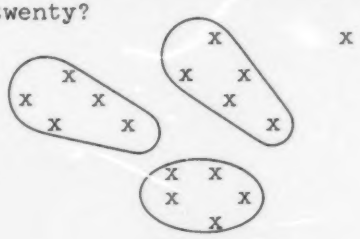
You have studied base seven numerals, so you now know that it is possible to express numbers in systems different from the decimal scale. Many persons think that the decimal system is used because the base ten is superior to other bases, or because the number ten has special properties. Earlier it was indicated that we probably use ten as a base because man has ten fingers. It was only natural for primitive people to count by making comparisons with their fingers. If man had had six or eight fingers, he might have learned to count by sixes or eights.

Our familiar decimal system of notation is superior to the Egyptian, Babylonian, and others because it uses the idea of place value and has a zero symbol, not because its base is ten. The Egyptian system was a tens system, but it lacked efficiency for other reasons.

Bases Five and Six

Our decimal system uses ten symbols. In the seven system you used only seven symbols, 0, 1, 2, 3, 4, 5, and 6. How many symbols would Eskimos use counting in base five? How many symbols would base six require? A little thought on the preceding questions should lead you to the correct answers. Can you suggest how many symbols are needed for base twenty?

The x's at the right are grouped in sets of five. How many groups of five are there? How many ones are left?



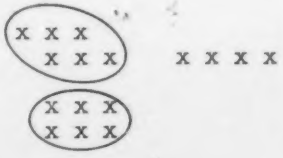
The decimal numeral for the number of x's in this diagram is 16. Using the symbols 0, 1, 2, 3, and 4, how would 16_{ten} be represented in base five numerals? An Eskimo, counting in base five, would think:

There are 3 groups of five and 1 more,

$$16_{ten} = (3 \times \text{five}) + (1 \times \text{one}),$$

$$16_{ten} = 31_{five}.$$

In the drawing at the right sixteen x's are grouped by sixes. How many groups of six are there? Are there any x's left? How would you write 16_{ten} in base six numerals?



There are 2 groups of six and 4 more,

$$16_{ten} = (2 \times \text{six}) + (4 \times \text{one}),$$

$$16_{ten} = 24_{six}.$$

Write sixteen x's. Enclose them in groups of four x's. Can you write the numeral 16_{ten} in base four numerals? How many groups of four are there? Remember, you cannot use the symbol "4" in base four. A table of the powers of four in decimal numerals is shown on the following page.

$(four^3)$	$(four^2)$	$(four^1)$	(one)
$(4 \times 4 \times 4)$	(4×4)	(4)	(1)
(64)	(16)	(4)	(1)

To write sixteen x's in base four we need (1 group of $four^2$) + (0 groups of four) + (0 ones). That is, $16_{ten} = 100_{four}$.

Exercises 1-6

1. Draw sixteen x's. Group the x's in sets of three.
   a. There are _____ groups of three and _____ left over.
   b. Are your answers to Part (a) both digits in the base three system? Why not?
   c. In sixteen x's there are (_____ groups of $three^2$) + (_____ groups of three) + (_____ left over).
   d. 16_{ten} = _____three.

2. Draw groups of x's to show the numbers represented by the following numerals. Then write the decimal numerals for these numbers.
   a. 23_{four} b. 15_{six} c. 102_{three} d. 21_{five}.

3. Write in base five notation the numbers from one through thirty. Start a table as shown below:

Base ten	0	1	2	3	4	5	6	7
Base five	0	1	?	?	?	?	?	?

4. a. How many threes are there in 20_{three}?
   b. How many fours are there in 20_{four}?
   c. How many fives are there in 20_{five}?
   d. How many sixes are there in 20_{six}?

5. Write the following in expanded notation. Then write the base ten numeral for each as shown in the example.

   Example: $102_{five} = (1 \times 25) + (0 \times 5) + (2 \times 1) = 27$

   a. 245_{six} c. 1002_{three}
   b. 412_{five} d. 1021_{four}

6. Write the following decimal numerals in bases six, five, four, and three. Remember the values of the powers for each of these bases. Note the example:

$$7_{ten} = 11_{six} = 12_{five} = 13_{four} = 21_{three}.$$

a. 11_{ten} b. 15_{ten}

c. 28_{ten} d. 36_{ten}

1-7. The Binary System.

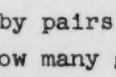
The use of binary notation in high speed computers is, of course, well known. The binary system is used for computers since there are only two digits, and an electric mechanism is either "on" or "off." The base two has the disadvantage that, while only two different digits are used, many more places are needed to express numbers in binary notation than in decimal, e.g.,

$$2000_{ten} = 11,111,010,000_{two}.$$

There is another base of special interest. The base two, or binary, system is used by some modern, high speed computing machines. These computers, sometimes incorrectly called "electronic brains," use the base two as we use base ten.

Historians tell of primitive people who used the binary system. Some Australian tribes still count by pairs, "one, two, two and one, two twos, two twos and one," and so on.

The binary system groups by pairs as is done with the three x's at the right. How many groups of two are shown? How many single x's are left? Three x's means 1 group of two and 1 one. In binary notation the numeral 3_{ten} is written 11_{two}.



Counting in the binary system starts as follows:

Decimal numerals	1	2	3	4	5	6	7	8	9	10
Binary numerals	1	10	11	100	101	110	111	?	?	?

How many symbols are needed for base two numerals? Notice that the numeral 101_{two} represents the number of fingers on one hand. What does 111_{two} mean?

$$111_{two} = (1 \times two^2) + (1 \times two^1) + (1 \times one) = 4 + 2 + 1 = 7_{ten}.$$

How would you write 8_{ten} in binary notation? How would you write 10_{ten} in binary notation? Compare this numeral with 101_{two}.

Modern high speed computers are electrically operated. A simple electric switch has only two positions, open (on) or closed (off). Computers operate on this principle. Because there are only two positions for each place, the computers use the binary system of notation.

We will use the drawing at the right to represent a computer. The four circles represent four lights on a panel, and each light represents one place in the binary system. When the current is flowing the light is on, shown in Figure 1-7b as ◉. A ◉ is represented by the symbol "1". When the current does not flow, the light is off, shown by ○ in Figure 1-7b. This is represented by the symbol "0". The panel in Figure 1-7b represents the binary numeral 1010_{two}. What decimal numeral is represented by this numeral? The table at the right shows the place values for the first five places in base two numerals.

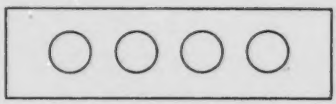


Figure 1-7a

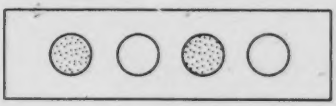


Figure 1-7b

two^4	two^3	two^2	two^1	one
2x2x2x2	2x2x2	2x2	2	1
16	8	4	2	1

$$1010_{two} = (1 \times two^3) + (0 \times two^2) + (1 \times two^1) + (0 \times one)$$
$$= (1 \times 8) + (0 \times 4) + (1 \times 2) + (0 \times 1)$$
$$= 10_{ten}.$$

Exercises 1-7

1. Make a counting chart in base two for the numbers from zero to thirty-three.

Base Ten	1	2	3	4	. . .	33
Base Two	1	10	11		. . .	

2. Copy and complete the addition chart for base two shown at the right. How many addition facts are there?

Addition, Base Two

+	0	1
0		
1		

3. Using the same form as in Exercise 2, make a multiplication chart for base two. How many multiplication facts are there? How do the tables compare? Does this make working with the binary system difficult or easy? Explain your answer.

4. Write the following binary numerals in expanded notation and then in base ten notation.

a. 111_{two} c. 10101_{two}

b. 1000_{two} d. 11000_{two}

e. 10100_{two}

5. Add these numbers which are expressed in binary notation. Check by expressing the numerals in the exercises, and in your answers, in decimal notation and adding the usual way.

a. 101_{two} b. 110_{two} c. 10110_{two} d. 10111_{two}

$\underline{\quad 10_{two}}$ $\underline{\quad 101_{two}}$ $\underline{\quad 11010_{two}}$ $\underline{\quad 11111_{two}}$

6. Subtract these base two numbers. Check your answers as you did in Exercise 5.

a. 111_{two} b. 110_{two} c. 1011_{two} d. 11001_{two}

$\underline{\quad 101_{two}}$ $\underline{\quad 11_{two}}$ $\underline{\quad 100_{two}}$ $\underline{\quad 10110_{two}}$

7. When people operate certain kinds of high speed computing machines, it is necessary to express numbers in the binary system. Change the following decimal numerals to base two notation:

a. 35 b. 128 c. 12 d. 100

Chapter 2

WHOLE NUMBERS

2-1. Counting Numbers.

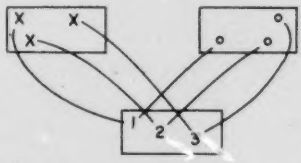
A seventh grade arithmetic course customarily begins by reviewing the arithmetic studied in earlier grades. Frequently the review consists mostly of drill problems. The present chapter illustrates a way in which arithmetic can be reviewed in such a way as to bring out some of the basic, unifying ideas of arithmetic without sacrificing practice on arithmetic skills.

The counting numbers are the numbers used to answer the question "How many?" Primitive man developed the idea of number by the practice of matching objects, or things, in one set with objects in another set. When a man's sheep left the fold in the morning he could put a stone in a pile as each sheep went out. When the sheep returned in the evening he took a stone out of the pile as a sheep went into the pen. If there were no stones left in the pile when the last sheep was in the pen he knew that all the sheep had returned. Similarly, in order to keep count of the number of wild animals he had killed he could make notches in a stick--one notch for each animal. If he were asked how many animals he had killed he could point to the notches in the stick. The man was saying that there were just as many animals killed as there were notches in the stick. The man was trying to answer the question "How many?" by making a one-to-one correspondence between the animals and the notches in the stick. He was also trying to answer the question "How many?" by making a one-to-one correspondence between the stones of the pile and the sheep of the flock. The one-to-one correspondence means that exactly one stone corresponded to each sheep and exactly one sheep corresponded to each stone. This says that the number of sheep was the same as the number of stones.

Some of us have learned the meaning of number in counting by using such one-to-one correspondences. We look at various sets of objects as in the figure. We see that there is a certain property that these sets possess. This property

may be described by saying that there are "just as many" marks in one set as in the other. A one-to-one correspondence between the sets can be shown by joining the marks with strings, or lines. Each mark is joined to a mark of the other set. No marks are left over in either set and no mark is used twice. The correspondence shows that there are "just as many" marks in one set as in the other but it does not tell us "how many" there are in terms of a number.

Fortunately we have a standard set which we can use to tell us "how many" there are in each set. It also can be used to tell us that there are "just as many" in one set as in the other. This standard set is the set of counting numbers represented by the numerals 1, 2, 3, 4, 5, ... In the figure each set of marks is put in a one-to-one correspondence with the set of numerals 1, 2, 3. The number of marks is the same as the number represented by the last numeral of the matching set. This kind of one-to-one correspondence between the marks and the set of numerals tells us that there are "just as many" in one set as in the other, and also tells us "how many" marks are in each set.



The method of using the counting numbers is such a natural one that the counting numbers are also called the "natural numbers." In this text we call them counting numbers. You may see them called "natural numbers" in other books.

Let us agree that our first counting number is 1. If we wish to talk about all the counting numbers and zero we call this set of numbers the "whole numbers."

2-2. Commutative Properties for Whole Numbers.

The commutative, associative, and distributive properties are basic concepts not only in arithmetic but also in algebra. They are not new to students. In fact, students have used them for a long time, but they probably have not had names for them and have not recognized when they were using them. It should be emphasized that these properties refer to operation on numbers, not to numbers themselves, and do not depend on the numeration system that is used.

If you have three apples in a basket and put in two more, then the number of apples in the basket is obtained by adding 2 to 3. You think of 3 + 2. If you started with two apples in the basket, and put in three more, then the number of apples in the basket is obtained by adding 3 to 2. You think of 2 + 3. In either case it is clear that there will be 5 apples in the basket. We may write 2 + 3 = 3 + 2.

The arithmetic teacher read two large numbers to be added. One boy did not understand what his teacher said when she read the first number. He wrote the second number and then asked her to repeat the first number. When she read it again, he wrote it below the second number instead of above it. If all the students do the addition correctly, will the boy find the same sum as the students who heard all the dictation the first time?

The boy wrote: $\begin{array}{r} 2437 \\ \underline{6254} \end{array}$ The others wrote: $\begin{array}{r} 6254 \\ \underline{2437} \end{array}$

We call this idea which was just described the commutative property of addition for whole numbers. It means that the order in which we add two numbers does not affect the sum. The word property is used here in the usual meaning of the word--it is something that belongs to the operation of addition:

3 added to 4 is 7 or $4 + 3 = 7$,
4 added to 3 is 7 or $3 + 4 = 7$.

Thus, we can write $4 + 3 = 3 + 4$. This checks the commutative property of addition for these two whole numbers.

The commutative property of addition for whole numbers may be stated as:

Property 1. If a and b represent whole numbers then

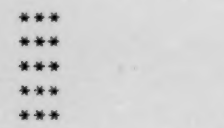
$$a + b = b + a.$$

In the above example a is 4 and b is 3.

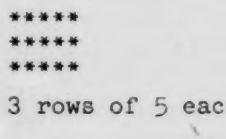
Multiplication is another operation which we perform on numbers. Is there a commutative property of multiplication? Let us see how to find the answer to the question.

Suppose we have five rows of chairs with 3 chairs in each row. Then, suppose we decide to change the arrangement to make

three rows with 5 chairs in each row. Will we need more chairs? Will we have any chairs which are not used in the second arrangement?



5 rows of 3 each: $5 \times 3 = 15$



3 rows of 5 each: $3 \times 5 = 15$

In learning the multiplication tables you learned that $7 \times 5 = 35$ and that $5 \times 7 = 35$. Similarly $9 \times 8 = 72$ and $8 \times 9 = 72$. When the two numbers are the same, the products are the same, regardless of which number is written first.

These examples indicate that there is a commutative property of multiplication. This commutative property of multiplication for whole numbers states that the product of two whole numbers is the same whether the first be multiplied by the second or the second be multiplied by the first. We state this as:

Property 2. If a and b represent whole numbers, then

$$a \times b = b \times a.$$

We can use this property to detect mistakes which we might make in multiplying one number by another. We found these products:

$$\begin{array}{r} 436 \\ \underline{125} \\ 2180 \\ 872 \\ \underline{436} \\ 54500 \end{array} \qquad \begin{array}{r} 125 \\ \underline{436} \\ 730 \\ 365 \\ \underline{600} \\ 63380 \end{array}$$

In this computation the commutative property shows that we have made at least one mistake. Find all the mistakes.

In both Property 1 and Property 2 we used letters to represent numbers. This idea of using letters to stand for any number whatsoever in stating general principles is a very useful part of mathematical language. Sometimes the letter x and the multiplication sign may be mistaken for each other, so we often use a raised dot, $\cdot$, to indicate multiplication. For example we can write $4 \cdot 3$ for 4×3 and $a \cdot b$ for $a \times b$.

Many symbols are used to simplify the writing of mathematics. Any symbol can be introduced and used if we first decide what the symbol is to mean and always use it to have that meaning. The use of the raised dot is a good example.

In mathematics we often say that one number is greater than another. To simplify writing the phrase "is greater than" we use the symbol $>$. So, to write "5 is greater than 3" we merely write $5 > 3$. To indicate that "a is greater than b" we write $a > b$. Similarly, we use the symbol $<$ to mean "is less than." Hence, we write $4 < 7$ for "4 is less than 7." Notice that each of these new symbols points toward the smaller of the two numbers being compared.

Sometimes we merely wish to note that two numbers are not equal. The symbol $\neq$ is used for "is not equal to." For examples, $5 \neq 3$ and $4 \neq 0$.

In comparing three numbers such as 3, 6, and 11, we may write $3 < 6 < 11$ or $11 > 6 > 3$. Note that the statement $3 < 6 < 11$ really stands for the two statements "3 is less than 6" and "6 is less than 11."

Exercises 2-2a

1. Indicate whether each statement is true or false:

   a. $6 + 4 = 4 + 6$

   b. $13_{four} + 32_{four} < 32_{four} + 13_{four}$

   c. $6 < 7 < 14$

   d. $1 + 5 = 5 + 1$

   e. $6 \cdot 5 = 5 \cdot 7$

   f. $6 + 3 = 4 + 5$

   g. $45 \cdot 36 < 36 \cdot 45$

   h. $5 + 4 > 5 + 3$

   i. $315 + 462 = 462 + 315$

   j. $5 > 3 > 10$

   k. $8 \div 2 = 2 \div 8$

   ℓ. $851 + 367 = 158 + 763$

   m. If $16 > 7$ and $7 > 5$ then $16 > 5$.

2. Add. Then use the commutative property to check addition.

   a. $\begin{array}{r} 465 \\ \underline{179} \end{array}$ b. $\begin{array}{r} 37461 \\ \underline{73135} \end{array}$ c. $\begin{array}{r} 73967 \\ \underline{81785} \end{array}$ d. $\begin{array}{r} 43_{seven} \\ \underline{32_{seven}} \end{array}$

3. Using the symbols $=$, $<$, and $>$, make the following true.

a. $7 + 4 \;?\; 4 + 7$
b. $12 \cdot 5 \;?\; 5 \cdot 11$
c. $23 \cdot 12 \;?\; 12 \cdot 32$
d. $3 \;?\; 6$
e. $16 \;?\; 9 \;?\; 3$
f. $(3 \cdot 2) + 5 \;?\; 5 + (3 \cdot 2)$
g. $8 - 3 \;?\; 9 - 3$
h. $86 \cdot 135 \;?\; 135 \cdot 86$
i. $24 \div 3 \;?\; 3 \div 24$
j. Given that a, b, and c are whole numbers: If $a > b$ and $b > c$, then $a \;?\; c$.

4. Multiply. Then use the commutative property to check the multiplication.

a. $\begin{array}{r} 36 \\ \underline{57} \end{array}$
b. $\begin{array}{r} 305 \\ \underline{84} \end{array}$
c. $\begin{array}{r} 476 \\ \underline{609} \end{array}$
d. $\begin{array}{r} 31_{\text{seven}} \\ \underline{25_{\text{seven}}} \end{array}$

5. Give the whole number or whole numbers which may be used in place of a to make the statements true.

a. $3 + a = 3 + 5$
b. $5 \cdot 7 = 7 \cdot a$
c. $2 \cdot a < 2 \cdot 1$
d. $3 \cdot a < 3 \cdot 2$
e. $132 + a = 46 + 132$
f. $2 + a < 2 + 7$
g. $7 \cdot 3 > a \cdot 5$
h. $a + 3 = 3 + a$

The commutative properties of addition and multiplication have been stated in symbolic form:

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a.$$

Notice how similar the statements are.

Do you think subtraction has the commutative property? To find out we must ask whether $a - b$ is equal to $b - a$ for all whole numbers a and b. If we can find at least one pair of whole numbers for which it is not true, then subtraction cannot have the commutative property. Is $6 - 9$ equal to $9 - 6$? No. In fact, $(9 - 6)$ is 3 and there is no whole number which is $(6 - 9)$.

Exercises 2-2b

1. Interchange the numbers in each of the following. In which ones is the result unchanged?

a. $1 + 2$	d. $4 - 5$	g. $5 - 4$
b. $6 + 8$	e. $12 + 3$	h. $3 \div 12$
c. $7 \cdot 9$	f. $9 \div 4$	i. $4 + 9$

2. Does division of whole numbers have the commutative property? Give an example which illustrates your answer.

3. Which of the following activities are commutative?

   a. To put on a hat and then a coat.
   b. To put on socks and then shoes.
   c. To pour red paint into blue paint.
   d. To close the hatch and dive the submarine.
   e. To put on your left shoe and then the right shoe.

4. We shall invent the operation "M" which shall mean to choose the larger of two numbers. If the numbers are the same we shall choose that one number. Is the operation commutative?

   Example: $3 \text{ M } 4 = 4$

5. Which of the defined operations below are commutative?

   a. "D" means to find the sum of the first and twice the second. Example: $3 \text{ D } 5 = 3 + (2 \cdot 5)$ or 13.
   b. "Z" means to find the sum of the first and the product of the first and the second.
      Example: $4 \text{ Z } 7 = 4 + (4 \cdot 7)$ or 32.
   c. "F" means to find the product of the first and one more than the second. Example: $8 \text{ F } 0 = 8 \cdot 1$ or 8.
   d. "Q" means to find three times the sum of the first and the second. Example: $8 \text{ Q } 5 = 3 \cdot (8 + 5)$ or 39.

6. List some activities which are commutative and some which are not commutative.

2-3. The Associative Property.

What is meant by $1 + 2 + 3$? Do we mean $(1 + 2) + 3$ in which we add 1 and 2 and then add 3 to the sum? Or do we mean $1 + (2 + 3)$ in which we add 2 and 3 and then add their sum to 1?

Or, does it make any difference? We have seen that the order in which two numbers are added does not affect the sum (commutative property of addition). Now we see that the way we group three numbers to add them does not affect the sum. For example,

$$(1 + 2) + 3 = 3 + 3 = 6$$

and $$1 + (2 + 3) = 1 + 5 = 6.$$

We call this idea of grouping the numbers differently without changing the sum the associative property of addition for whole numbers. This property may be used to make addition easier if the sum of one pair of three numbers is easier to find than the sum of another pair. If you are asked to add $12 + 4 + 2$ you might first add 12 and 4 and then add 2 to 16. Or you might think of first adding 4 and 2 and then adding 6 to 12. If we add each of the following by grouping the numbers differently we will be showing applications of the associative property.

$$7 + 9 + 11 = 7 + (9 + 11) = 7 + 20 = 27$$

$$12 + 7 + 33 = 12 + (7 + 33) = 12 + 40 = 52$$

$$97 + 53 + 100 = (97 + 53) + 100 = 150 + 100 = 250$$

The associative property can be used in finding the sum of 12 and 7. Perhaps you have always used it but did not call it by name. Notice how it can be used:

$$12 + 7 = (10 + 2) + 7 = 10 + (2 + 7) = 19.$$

Just as we stated the commutative property of addition, we now state the associative property of addition.

Property 3. If a, b and c represent any whole numbers

$$(a + b) + c = a + (b + c).$$

What is meant by $2 \cdot 5 \cdot 4$? Do we mean $(2 \cdot 5) \cdot 4$ in which we first multiply 2 by 5 and then multiply 10 by 4, or do we mean $2 \cdot (5 \cdot 4)$ in which we first multiply 5 by 4 and then multiply 2 by 20? Both give the same answer and we conclude that we can give either meaning to $2 \cdot 5 \cdot 4$. This is true for any whole numbers.

Property 4. If a, b, and c represent any whole numbers,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

This is the symbolic statement of the associative property of multiplication for whole numbers.

In everyday life we speak of "adding" or combining several things. Whether such combinations have the associative property depends on the things we combine. Is (gasoline + fire) + water the same as gasoline + (fire + water)?

The commutative property of addition means we may change the order of any two numbers without affecting the sum. The associative property means that we may group numbers in pairs for the purpose of adding pairs of them without affecting the sum. Just as there is a commutative property for addition and multiplication, we might expect the associative property to belong to both operations.

Sometimes it is convenient to rearrange the order of the numbers which are to be added, or multiplied, in order to make the operation easier. This may be done by use of the commutative property. Then the addition or multiplication can be performed by grouping the numbers according to the associative property. The following examples are illustrations of the uses of both properties in the same problem.

$17 + (19 + 13) = 17 + (13 + 19) = (17 + 13) + 19 = 30 + 19 = 49$

$50 \cdot (17 \cdot 4) = 50 \cdot (4 \cdot 17) = (50 \cdot 4) \cdot 17 = 200 \cdot 17 = 3400$

Is there an associative property for subtraction? Perhaps we can answer the question by considering just one example. We try $10 - (6 - 4)$ which is $10 - 2$ or 8. But $(10 - 6) - 4 = 0$, so that $10 - (6 - 4)$ is not equal to $(10 - 6) - 4$. This shows that subtraction does not have the associative property. At first you may think that one example is not enough and that the property might hold if we used some other numbers. But, if the associative property is to hold for subtraction then it must hold for all whole numbers. Hence, by showing one set of three whole numbers for which the property is not true we know that it cannot be a property for <u>all</u> whole numbers.

Do you think the associative property holds for division? What does $16 \div 4 \div 2$ mean? We cannot tell. It may mean $(16 \div 4) \div 2$, or it may mean $16 \div (4 \div 2)$. The first of these

equals 2 and the second equals 8, so they are not equal to each other. This shows that division does not have the associative property.

These remarks about subtraction and division show us also that expressions like $10 - 6 - 4$ and $16 \div 4 \div 2$ do not have any meaning. Of course, the expressions, $(10 - 6) - 4$ and $10 - (6 - 4)$, do have meanings and they are different. Also, $(16 \div 4) \div 2$ and $16 \div (4 \div 2)$ make sense, but their meanings are different.

Exercises 2-3

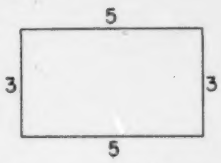
1. Example: $(4 + 3) + 2 = 4 + (3 + 2)$.
   Here, $(4 + 3) + 2 = 7 + 2 = 9$, and $4 + (3 + 2) = 4 + 5 = 9$.
   This illustrates the associative property of addition.
   Show that the following are true in the above way. State the property illustrated in each problem.
   a. $(4 + 7) + 2 = 4 + (7 + 2)$
   b. $8 + (6 + 3) = (8 + 6) + 3$
   c. $46 + (73 + 98) = (46 + 73) + 98$
   d. $(6 \cdot 5) \cdot 9 = 6 \cdot (5 \cdot 9)$
   e. $(21 + 5) + 4 = 21 + (5 + 4)$
   f. $(9 \cdot 7) \cdot 8 = 9 \cdot (7 \cdot 8)$
   g. $436 + (476 + 1) = (436 + 476) + 1$
   h. $(57 \cdot 80) \cdot 75 = 57 \cdot (80 \cdot 75)$

2. a. Does $(10 - 7) - 2$ equal $10 - (7 - 2)$?
   b. Does $18 - (5 - 2)$ equal $(18 - 5) - 2$?
   c. What generalization can you make regarding the associative property of subtraction?

3. a. Does $(32 \div 8) \div 2$ equal $32 \div (8 \div 2)$?
   b. Does $(60 \div 30) \div 2$ equal $60 \div (30 \div 2)$?
   c. Place parentheses in $75 \div 15 \div 5$ so that it will equal 1.
   d. Place parentheses in $75 \div 15 \div 5$ so that it will equal 25.
   e. Place parentheses in $80 \div 20 \div 2$ so that it will equal 8.
   f. Place parentheses in $80 \div 20 \div 2$ so that it will equal 2.
   g. What generalization can be made concerning the associative property and division?

2-4. The Distributive Property.

In finding the perimeter of the top of a desk one pupil measured the length of each side in feet and found the measurements as shown in the diagram. Then he found the perimeter in feet by finding the sum $5 + 3 + 5 + 3 = 16$. Another pupil said he thought that this was all right but that it was more work than necessary. He said he would add 5 and 3 and multiply their sum by 2. Will this give the same answer? A third pupil said she thought it would be better to multiply 5 by 2 and 3 by 2 and then add these two products. The second and third pupils may not have known the name of the principle they were using but it is useful and important. It is called the distributive property. In terms of the pupils' problem it states simply that



$$2 \cdot (5 + 3) = (2 \cdot 5) + (2 \cdot 3)$$

and $$2 \cdot (5 + 3) = (2 \cdot 8).$$

Eight girls and four boys are planning a skating party. Then, each girl invites another girl and each boy invites another boy. The original number of girls has been doubled. The original number of boys has been doubled. Has the total number of children been doubled or not? Let us see. In all there will be $(2 \cdot 8)$ girls and $(2 \cdot 4)$ boys or a total of $(2 \cdot 8) + (2 \cdot 4) = 24$ children at the party. Let us look at this another way. When the party was planned, there were $(8 + 4) = 12$ children. The final number of children is $2 \cdot (8 + 4)$ or $2 \cdot 12$. We have seen that $(2 \cdot 8) + (2 \cdot 4) = 16 + 8 = 24$ and $2 \cdot (8 + 4) = 2 \cdot 12 = 24$. So we can write $(2 \cdot 8) + (2 \cdot 4) = 2 \cdot (8 + 4)$.

You have been using this property in many ways for a long time. Consider, for example, $3 \cdot 13$ or $\begin{array}{r} 13 \\ \times\ 3 \\ \hline 39 \end{array}$. You were really using the distributive property because:

$$3 \cdot 13 = 3 \cdot (10 + 3) = (3 \cdot 10) + (3 \cdot 3) = 30 + 9 = 39.$$

Let us see how you use the distributive property in finding the product $9 \cdot 36$. You probably perform the multiplication about as follows:

$$\begin{array}{r} 36 \\ \times\ 9 \\ \hline 324 \end{array} \qquad \text{or} \qquad \begin{array}{rl} 36 & \\ \times\ 9 & \\ \hline 54 & (9 \times 6) \\ 270 & (9 \times 30) \\ \hline 324 & \end{array}$$

Do you see that the left example is a short way of doing the problem? You were really using the distributive property:

$$\begin{aligned} 9 \cdot 36 &= 9 \cdot (30 + 6) \\ &= (9 \cdot 30) + (9 \cdot 6) \quad \text{distributive property} \\ &= 270 + 54 \\ &= 324. \end{aligned}$$

The distributive property is also important in operations involving fractions. Let us find the product of 8 and $12\frac{1}{4}$. First, recall that $12\frac{1}{4}$ means $12 + \frac{1}{4}$. Then

$$\begin{aligned} 8 \cdot 12\tfrac{1}{4} &= 8 \cdot (12 + \tfrac{1}{4}) \\ &= (8 \cdot 12) + (8 \cdot \tfrac{1}{4}) = 96 + 2 \\ &= 98. \end{aligned}$$

The distributive property is:

Property 5. If a, b, and c are any whole numbers then

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

The distributive property is the only property of the three we have studied in this chapter which involves two operations, namely, addition and multiplication. This does not mean that any problem which involves these two operations is performed by using the distributive property. For example, $(3 \cdot 5) + 14$ means that the product of 3 and 5 must be found and then 14 added to the product: $(3 \cdot 5) + 14 = 15 + 14 = 29$.
However, $3 \cdot (5 + 14) = (3 \cdot 5) + (3 \cdot 14) = 15 + 42 = 57$.

The commutative property of multiplication permits us to write $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$. Let us see why.

First, $(b + c) \cdot a = a \cdot (b + c)$, commutative property

and $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, distributive property.

Therefore, $(b + c) \cdot a = (a \cdot b) + (a \cdot c)$.

Also, $(a \cdot b) = (b \cdot a)$ commutative property

$(a \cdot c) = (c \cdot a)$ commutative property.

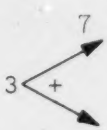
Hence, $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$.

This justifies the multiplication of $12\frac{1}{4}$ by 8 in the form

$$12\tfrac{1}{4} \cdot 8 = (12 + \tfrac{1}{4}) \cdot 8 = (12 \cdot 8) + (\tfrac{1}{4} \cdot 8)$$
$$= 96 + 2 = 98.$$

Some of you may wish to use a sketch to help you remember that the first factor is distributed over all the numbers being added in the second factor. One such sketch is illustrated here. For example, consider the product $3 \cdot (7 + 9)$.

Sketch:



$= 21 + 27 = 48.$

Note that denotes that the 3 multiples both the 7 and the 9 and that these products are then added. The arrows always point to the numbers that are to be added. Another example might be:

$$6 \cdot (5 + 8)$$

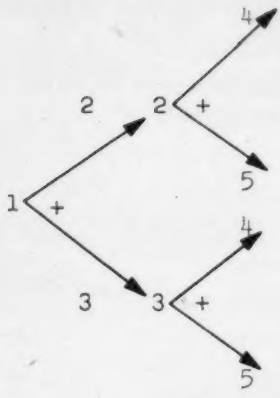
Sketch:

$= 30 + 48 = 78$ or

$= 78$

Another example.

$$(2 + 3) \cdot (4 + 5)$$



$= (8 + 10) + (12 + 15) = 45$

Exercises 2-4

1. Use the sketch method illustrated above to do the indicated operations.

   a. $5 \cdot (6 + 4)$
   b. $3 \cdot (9 + 6)$
   c. $12 \cdot (6 + 7)$
   d. $9 \cdot (13 + 17)$
   e. $(6 + 4) \cdot (8 + 7)$
   f. $(20 + 7) \cdot (10 + 4)$

2. Show that the following are true by doing the indicated operations. Example: $3 \cdot (4 + 3) = (3 \cdot 4) + (3 \cdot 3)$.

$$3 \cdot (4 + 3) = 3 \cdot 7 = 21$$

$$(3 \cdot 4) + (3 \cdot 3) = 12 + 9 = 21$$

   a. $4 \cdot (7 + 5) = (4 \cdot 7) + (4 \cdot 5)$
   b. $(3 \cdot 6) + (4 \cdot 6) = 6 \cdot (3 + 4)$
   c. $(8 \cdot 6) + (7 \cdot 6) = (8 + 7) \cdot 6$
   d. $23 \cdot (2 + 3) = (23 \cdot 2) + (23 \cdot 3)$
   e. $11 \cdot (3 + 4) = (11 \cdot 3) + (11 \cdot 4)$
   f. $(6 \cdot 5) + (6 \cdot 3) = 6 \cdot (5 + 3)$
   g. $2 \cdot (16 + 8) = (2 \cdot 16) + (2 \cdot 8)$
   h. $12 \cdot (5 + \frac{1}{4}) = (12 \cdot 5) + (12 \cdot \frac{1}{4})$
   i. $(67 \cdot 48) + (67 \cdot 52) = 67 \cdot (48 + 52)$
   j. $(72 \cdot \frac{1}{2}) + (\frac{1}{2} \cdot 72) = 72 \cdot (\frac{1}{2} + \frac{1}{2})$.

3. Make each of the following a true statement illustrating the distributive property.

   a. $3 \cdot (4 + \quad) = (3 \cdot 4) + (3 \cdot 3)$
   b. $2 \cdot (\quad + 5) = (2 \cdot 4) + (\quad \cdot 5)$
   c. $13 \cdot (6 + 4) = (13 \cdot \quad) + (13 \cdot \quad)$
   d. $(2 \cdot 7) + (3 \cdot \quad) = (\quad) \cdot 7$
   e. $(\quad \cdot 4) + (\quad \cdot 4) = (6 + 7) \cdot (\quad)$.

4. Using the distributive property rewrite each of the following.

   Examples: 1. $5 \cdot (2 + 3) = (5 \cdot 2) + (5 \cdot 3)$
   2. $(6 \cdot 4) + (6 \cdot 3) = 6 \cdot (4 + 3)$

   a. $(9 \cdot 8) + (9 \cdot 2)$
   b. $8 \cdot (14 + 17)$
   c. $12 \cdot (5 + 7)$
   d. $(13 + 27) \cdot 6$
   e. $15 \cdot (6 + 13)$
   f. $(5 \cdot 12) + (4 \cdot 12)$

5. Using the idea of the distributive property we can rewrite, for example:

   (1) $10 + 15$ as $(5 \cdot 2) + (5 \cdot 3)$ or $5 \cdot (2 + 3)$
   (2) $15 + 21$ as $(3 \cdot 5) + (3 \cdot 7)$ or $3 \cdot (5 + 7)$

   Use the distributive property to rewrite the following in a similar way.

   a. $35 + 40$
   b. $12 + 15$
   c. $55 + 10$
   d. $27 + 51$
   e. $100 + 115$
   f. $30 + 21$

6. Which of the following are true?

   a. $3 + (4 \cdot 2) = (3 + 4) \cdot (3 + 2)$
   b. $3 \cdot (4 - 2) = (3 \cdot 4) - (3 \cdot 2)$
   c. $(4 + 6) \cdot 2 = (4 \cdot 2) + (6 \cdot 2)$
   d. $(4 + 6) \div 2 = (4 \div 2) + (6 \div 2)$
   e. $3 + (4 \cdot 2) = (3 \cdot 4) + (3 \cdot 2)$

2-5. Inverse Operations.

Subtraction is often taught quite separately from addition, and the same is true for division and multiplication. One aim of this section is to clarify the relations between these operations.

Often we do something and then we undo it. We open the door; we shut the door. We open the window; we close the window. One operation is the inverse of the other.

The inverse of putting on your coat is taking off your coat. The inverse operation of division is multiplication. The inverse operation of addition is subtraction.

Suppose you have \$220 in the bank and you add \$10 to it. Then you have \$220 + \$10 = \$230. Now undo this by drawing out \$10. The amount that remains is \$230 - \$10 = \$220. The athletic fund at your school might have \$1800 in the bank and after a game have \$300 more. Then the fund has \$1800 + \$300 or \$2100 in it. But the team needs new uniforms which cost \$300 so \$300 is withdrawn to pay for them. The amount left is \$2100 - \$300, or \$1800. These operations undo each other. Subtraction is the inverse of addition.

Of course, we could express this idea in more general terms. Let x represent the number of dollars originally in the bank. If the amount we deposit is b, then $x + b = a$, where a represents the number of dollars we now have in the bank. How shall we undo this operation? From the number of dollars represented by a, we subtract the number of dollars withdrawn, represented by b, and we have the number represented by x. We write $x = a - b$.

You use the idea of inverse operation when you use addition in checking subtraction. For example:

$$\begin{array}{r} 203 \\ -\ 96 \\ \hline 107 \end{array} \qquad \begin{array}{r} a \\ -\ b \\ \hline x \end{array} \qquad \text{check:} \quad \begin{array}{r} 107 \\ +\ 96 \\ \hline 203 \end{array} \qquad \begin{array}{r} x \\ +\ b \\ \hline a \end{array}$$

You also use the idea of inverse operation when you use multiplication to check division. For example:

$$\begin{array}{r} 18 \\ 16\overline{)288} \\ \underline{160} \\ 128 \\ \underline{128} \\ 0 \end{array}$$

check:

$$\begin{array}{r} 16 \\ \times\ 18 \\ \hline 128 \\ \underline{160} \\ 288 \end{array}$$

or $288 \div 16 = 18$ check: $288 = 18 \times 16$.

Notice that if a and b are whole numbers, and if $a > b$, then there is a whole number x so that $b + x = a$. Examples: If a is 17 and b is 10, then x is the whole number 7 so that $10 + 7 = 17$; if a is 41 and b is 35, then x is the whole number 6 so that $35 + 6 = 41$. When a is greater than b it is always possible to find x so that $a = b + x$. Can you make the same generalization if the above operation $b + x = a$, is changed to multiplication, $b \cdot x = a$? If you substitute 2 for b and 3 for a you will see that there is no <u>whole</u> number that can be substituted for x such that $2 \cdot x = 3$. If one substitutes certain numbers--for example, if $a = 20$ and $b = 4$--then there is a whole number that can be substituted for x such that $4 \cdot x = 20$. In this example x must represent 5, since $4 \cdot 5 = 20$. We get the 5 by dividing 20 by 4. Also:

If b is 6 and a is 24 then x must be 4 since $6 \cdot 4 = 24$.
If b is 5 and a is 40 then x must be 8 since $5 \cdot 8 = 40$.
If b is 3 and a is 30 then x must be 10 since $3 \cdot 10 = 30$.

In each example the number for x is found by dividing the number represented by a by the number represented by b. In general, if there is a counting number x that can be multiplied by a counting number b to get counting number a, then this number x can be found by dividing a by b. We write this as $b \cdot x = a$. We multiply x by b to obtain a. To undo the operation we must perform the inverse operation which means that we must divide a by b to obtain x: $b\overline{)a}^{x}$. The inverse operation of multiplying by b is dividing by b.

Exercises 2-5

1. Select the words or phrases that describe operations that have an inverse. An operation followed by its inverse returns to the original situation.

   a. Picking up the pencil. (Remember, "not picking up the pencil" is not an inverse operation. "Not picking up the pencil" does not undo the operation of picking up the pencil.)
   b. Put on your hat.
   c. Getting into a car.
   d. Extend your hand.
   e. Multiply.
   f. Build.
   g. Smell the rose.
   h. Step forward.
   i. Jump from a flying airplane.
   j. Addition.
   k. Cutting off a dog's tail.
   l. Subtraction.
   m. Looking at the stars.
   n. Talking.
   o. Taking a tire off a car.

2. Write the inverse operation to each of those operations selected in Exercise 1.

3. Perform the indicated operation and check by the inverse operation.

   Subtract in (a) to (f)

   a. $\begin{array}{r} 89231 \\ -\ 42760 \end{array}$

   b. $\begin{array}{r} \$805.06 \\ -\ \$297.96 \end{array}$

   c. $\begin{array}{r} 803 \text{ ft.} \\ -\ 297 \text{ ft.} \end{array}$

   d. $\begin{array}{r} \$4302.14 \\ -\ \$2889.36 \end{array}$

   e. $\begin{array}{r} \$8000.02 \\ -\ \$6898.98 \end{array}$

   f. $\begin{array}{r} \$10040.50 \\ -\ \$\ 8697.83 \end{array}$

   g. $29\overline{)25404}$

   h. $38\overline{)37506}$

   i. $27\overline{)21546}$

   j. $19\overline{)13243}$

k. One hundred twenty minus eighty-seven.

l. The sum of six hundred forty-seven and eight hundred twenty-nine.

m. The difference between eighty-nine and twenty-one.

n. Seventy-six plus sixty-seven.

o. The product of three hundred six and one hundred ninety.

4. Find, if possible, a whole number which can be used for x in each of the following to make it a true statement. If there is no whole number that can be used for x, then say there is none.

a. $9 + x = 14$
b. $x + 9 = 14$
c. $x + 1 = 2$
d. $4 + x = 11$
e. $10 + x = 7$
f. $5 + x = 5$
g. $10 = x + 2$
h. $x = 9 - 5$
i. $x = 11 - 8$
j. $8 + x = 11$
k. $6 + x = 3$
l. $x = 13 - 6$
m. $3 + x = x + 3$
n. $3 \cdot x = 12$
o. $4 \cdot x = 20$
p. $x = 20 \div 4$
q. $2 \cdot x = 18$
r. $x = 18 \div 2$
s. $5 \cdot x = 30$
t. $2 \cdot x = 0$
u. $x = 0 \div 2$
v. $9 \cdot x = 0$
w. $x = 0 \div 9$
x. $3 \cdot x = 3$
y. $x = 3 \div 3$
z. $11 \cdot x = 11$

5. a. If one bookcase will hold 128 books and another 109 books, how many more books does the former hold?

b. A theatre sold 4789 tickets one month and 6781 tickets the next month. How many more people came to the theatre the second month than came the first month?

c. If one building has 900 windows and another building 811 windows, how many more windows does the first building contain?

d. The population of a town was 19,891 people. Five years later the population was 39,110 people. What was the increase of population for the five years?

e. If one truck can carry 2099 boxes, how many boxes can 79 similar trucks carry?

f. How many racks are needed to store 208 chairs, if each rack holds 16 chairs?

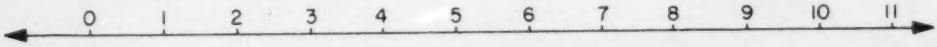
g. At a party there were 288 pieces of candy. If there were 48 children at the party, how many pieces of candy could each have?

h. A girl scout troop has 29 members. Each member is to sell boxes of cookies. If the troop has 580 boxes to sell, how many boxes will each girl have to sell in order to sell all of them?

2-6. Betweenness and the Number Line.

A graphical representation of the system of counting numbers makes it easier to see some of the structure of this system. At the same time, it is the first link in a chain which will later connect arithmetic and geometry.

How whole numbers are related may be shown with a picture. Select some point on a line as below and label it zero (0).



Label the first dot to the right of zero the first counting number and each dot after that to the right the succeeding counting number. This picture is often referred to as The Number Line. Any whole number is smaller than any of the numbers on the right side of it and greater than any of the numbers on its left. For example, 3 is less than 5 and greater than 2. This may be written $2 < 3 < 5$, since 2 is less than 3 and 3 is less than 5. With the number line we can also determine how many whole numbers there are between any two whole numbers. For example, to find how many whole numbers there are between 6 and 11 we can look at the picture and count them. We see four of them, 7, 8, 9, and 10.

Exercises 2-6

1. How many whole numbers are there between:

a. 7 and 25
b. 3 and 25
c. 20 and 25
d. 17 and 25
e. 25 and 25
f. 28 and 25
g. 26 and 25
h. $11\frac{1}{4}$ and 25

1. If a and b are whole numbers, and a > b, is the number of whole numbers between a and b :

   (1) b - a ? (3) a - (b + 1) ?
   (2) (a - 1) + b ? (4) (a - b) + 1 ?

2. What is the whole number midway between:

   a. 7 and 13
   b. 9 and 13
   c. 20 and 28
   d. 10 and 50
   e. 17 and 19
   f. 17 and 27
   g. 12 and 20
   h. 12 and 6

3. Which of the following pairs of whole numbers have a whole number midway between them?

   a. 6, 8
   b. 6, 10
   c. 8, 18
   d. 8, 13
   e. 7, 12
   f. 26, 33
   g. 9, 17
   h. 19, 36
   i. a, b if a and b are even whole numbers
   j. a, b if a and b are odd whole numbers
   k. a, b if a is odd and b is even

4. The whole numbers a, b and c are so located on The Number Line that b is between a and c, and c > b.

   a. Is c > a? Explain with a number line.
   b. Is b > a? Explain with a number line.
   c. Is b < c? Explain with words.

5. The whole numbers a, b, c and d are so located on The Number Line that b is between a and c and a is between b and d. What relation, if any, is there among b, c, and d?

2-7. The Number One.

The number one is a special number in several ways. One is the smallest of our counting numbers. We may build any number, no matter how large, by beginning with 1 and adding 1's until we have reached the desired number. For example, to obtain the number five, we can begin with our special number 1 and repeat the addition of 1. 1 + 1 = 2, 2 + 1 = 3, 3 + 1 = 4, 4 + 1 = 5. There is no largest counting number.

Also, it will be observed that for any of the counting numbers (1, 2, 3, ...) which we may select, we get the next larger counting number by adding 1. This may seem obvious to you because you have used the numbers so many times. In some of the fundamental operations we do not get the next counting number by operations using only the number 1; e.g., $3 \cdot 1 = 3$, $3 - 1 = 2$. In one case we do not even get a counting number. Observe what happens when we use the operation of subtraction: $1 - 1 = 0$. Zero is not a counting number.

In multiplication if we wish to obtain a different numeral for a number, we can multiply by a selected form of the special number 1. In this way we may get a different numeral, but it represents the same number. You may recall that in rewriting 4 as $\frac{8}{2}$, you were simply multiplying 4 by $\frac{2}{2}$. Of course, $\frac{2}{2}$ is our special number 1. Multiply $\frac{1}{3}$ by $\frac{3}{3}$ and get $\frac{3}{9}$; multiply $\frac{4}{5}$ by $\frac{2}{2}$ and get $\frac{8}{10}$. These are examples of multiplying by the number 1 in selected forms $\frac{3}{3}$, and $\frac{2}{2}$. This means that the new fractions are different in form from the original ones but they still represent the same number. The special number one when used as a multiplier makes the product identical with the multiplicand. Because the product of any counting number and one is the original counting number, the number 1 is called the "identity element" for multiplication.

Since division is the inverse operation of multiplication, is the number one also special in division? What happens if we divide any counting number by one? We do obtain the same counting number. But if we divide 1 by a counting number we do not get the counting number. For this reason we cannot say that the number one is the identity element for division. A counting number multiplied by 1 is the same number as 1 multiplied by the counting number but the same thing cannot be said for division. If we let C represent any counting number we can express these multiplication and division operations using the number 1 in the following ways.

$$C \cdot 1 = 1 \cdot C;$$
$$C \div 1 = C;$$
$$C \div C = 1;$$
$$1 \div C \neq C \text{ if } C \neq 1.$$

We have learned to use 10^2 to mean $10 \cdot 10$; 10^3 to mean $10 \cdot 10 \cdot 10$; 10^6 to mean $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$. The "2," "3," "6" are called exponents. The exponents are small, but the numbers represented by 10^2, 10^3 and 10^6 are very large. If we use 1 in place of 10 this is not true. For $1^2 = 1 \cdot 1$; $1^3 = 1 \cdot 1 \cdot 1$; $1^6 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ and these are still the number 1. In fact 1^2 or 1^{200} or 1^{3056} is still 1.

When we say 1^2 or 1^{200} is really only 1, we are just giving different names to the same thing. It is true that the number represented by each of these expressions is 1. Can you think of other such combinations of symbols which represent 1? What number is represented by 5 - 4? X - IX?

Our discussion of the number one may be summarized briefly in the mathematical sentences below. Can you translate them into words? The letter C here represents any counting number.

a. $C = 1$ or $(1 + 1)$ or $(1 + 1 + 1)$ or ... etc.
b. $1 \cdot C = C$
c. $C \div 1 = C$
d. $C \div C = 1$
e. $1^C = 1$

Exercises 2-7

1. From the following symbols, select those that represent the number 1:

a. I	e. $1 + 0$	i. 1	m. $\frac{1}{2} + \frac{1}{2}$
b. $\frac{4}{4}$	f. $1 \cdot 2$	j. $1 \cdot 0$	n. $1 \cdot 100$
c. $5 - 4$	g. $\frac{6}{4}$	k. $\frac{200}{200}$	o. $\frac{8 - 1}{12 - 5}$
d. $1 - 0$	h. $\frac{4}{1}$	l. 1^{10}	p. $\frac{2 - 1}{1}$

2. Copy and fill in the blanks:

a. $100 \cdot 1 =$ _____	d. $1 \cdot \frac{2}{3} \cdot 1 =$ _____
b. $10 \cdot 1 \cdot 1 \cdot 1 =$ _____	e. $0 \cdot 1 =$ _____
c. $\frac{14}{1} =$ _____	f. $1 \cdot 0 =$ _____

3. Can you get any counting number by the repeated addition or subtraction of 1 to or from any other counting number? Give an example to support your conclusion.

4. By the above process can you get a number that is not a counting number? Give an example to support your conclusion.

5. Robert said, "The counting numbers are not closed under the subtraction of ones but they are closed under the addition of ones." Show by an example what Robert meant.

6. Perform the indicated operations:

a. $(4 - 3) \overline{)876429}$

b. $1\overline{)976538}$

c. $897638 \cdot (5 - 4)$

d. $896758 \cdot \frac{4}{4}$

e. $3479 \cdot 1^{110}$

f. $97 \cdot x^6$ (if x is 1)

g. $1^7 \cdot (489 \div 489)$

h. $\frac{8}{8} \cdot 1^5 \div 1^4$

2-8. The Number Zero.

Arithmetic operations involving the number zero are a frequent source of confusion, even for many adults. The object of this section is to clarify this situation.

Another special number is zero. Occasionally you will hear it called by other names, such as "naught." When you answer a telephone a voice may say, "Is this 'one eight oh three'?" Of course, the caller is not referring to the letter "o," and all of us understand that he means "one eight zero three."

Although zero is not included in the counting numbers, it is considered as one of the whole numbers. Most of the time we use it according to rules of the counting numbers, and in a sense it is used to count. If you withdraw all your money from the bank, you can express your bank balance with this special number zero. If you have answered no questions correctly, your test score may be zero. If there are no chalkboard erasers in the classroom, the number of erasers may be expressed by zero. In all these cases, no money in the bank, no correctly answered questions and no erasers, the zero indicated that there are no objects or elements in the set of objects being discussed. If there are no elements in the set, we call it an empty set.

The number zero is the number of elements in the empty set. In this sense, some persons say that zero means "not any." Others say it means "nothing" because there is nothing in the set. As we shall see, these are rather confused and limited concepts of zero.

On a very cold morning Paul was asked the temperature. After looking at the thermometer he replied, "zero." Did he mean there was "not any"? Did he mean "nothing"? No, he meant the top of the mercury was at a specific point on the scale called zero. Fred had an altimeter in his car so he could check the altitude as they drove in the Rocky Mountains. On one vacation trip they drove to the Salton Sea. On the way down Fred exclaimed, "Look, the altitude is zero!" When the altimeter indicates zero, it does not mean there is "nothing," it means we are at a specific altitude which is called zero. It is just as specific and real as an altitude of 999 feet.

We noticed that the sum of a counting number and one is always the next larger counting number. The sum of a counting number and zero is always the original counting number. For example, $4 + 0 = 4$. We might express this fact in symbols $C + 0 = C$ where C is any counting number. Or we might express the fact by saying that zero is the "identity element" for addition.

The difference between the same two natural numbers is the special number zero. For example, $4 - 4 = 0$. Did you notice that in this subtraction operation you do not get a counting number? To put the idea in more elegant language, we would say that the set of counting numbers is not closed under subtraction.

Let us look at the special number zero under the operation of multiplication. What could $3 \cdot 0$ mean? We might think of the number of chairs in 3 rooms if each room contains zero chairs. Thus, any number of rooms containing zero chairs would have a total of zero chairs. We might express this idea in symbols by writing $C \cdot 0 = 0$, where C is any counting number.

The product $0 \cdot 3$ is even more difficult to explain. But we do know by the commutative property for multiplication that $3 \cdot 0 = 0 \cdot 3$. We have seen that $3 \cdot 0 = 0$. Therefore, we must have $0 \cdot 3 = 0$ as we wish the commutative property for multiplication to be true for <u>all</u> whole numbers. If a represents any whole number, we may express this by writing $a \cdot 0 = 0 \cdot a = 0$. If a is zero we must have $0 \cdot 0 = 0$.

There is a very important principle expressed in the above symbols, but it may not be seen at the first glance. Did you observe that if the product of two or more whole numbers is zero, then one of the numbers must be zero? For example, $4 \cdot 5 \cdot 0 = 0$. In mathematics you will use this fact frequently.

Let us see if zero follows the rules for division of counting numbers.

What could zero divided by 3 mean? If we have a room with zero chairs and divide the room into three parts, it could mean the number of chairs in each part of the room. With this meaning, $0 \div 3$ should be 0. If $3\overline{)0}$ then 0×3 should be zero, by the inverse operation. Does this agree with the definition of multiplication by zero?

Occasionally students forget that the division of zero by a counting number is always zero and never a counting number. For example, $\frac{0}{7} = 0$, $\frac{0}{7} \neq 7$.

If $\frac{0}{7} = 0$, what is $\frac{7}{0}$? Is $0\overline{)7}$ a counting number? Let us assume that $0\overline{)7}$ is equal to some number represented by N. This means that 7 is equal to zero times some number N. $(7 = 0 \times N)$. The product of any number by zero is zero; therefore, there is no number N that will equal $0\overline{)7}$. In more elegant language, we may say that $\frac{7}{0}$ is not the name of any counting number or zero. Therefore, we cannot perform this operation. <u>We cannot divide a counting number by zero</u>.

Could we divide zero by zero? In symbols the question is "$\frac{0}{0} = ?$". Or $0\overline{)0}^{?}$. If $0\overline{)0}$ equals some number n then by our definition of multiplication, $0 \times n = 0$. What numbers could

replace n ? Could n be 3? Of course, n could be any counting number or zero. Since $0\overline{)0}$ could be any whole number, the symbol $\frac{0}{0}$ has too many meanings. Therefore, we should remember that we cannot divide either a counting number or zero by zero.

Mary summarized the operations with the special number zero in these symbols. State them in words if u and w represent any whole numbers and C represents any counting number.

a. $w + 0 = w$
b. $0 + w = w$
c. $w - 0 = w$
d. $0 \cdot w = 0$
e. $w \cdot 0 = 0$
f. If $u \cdot w = 0$, then either u or w is zero or both are zero.
g. $0 \div C = 0$
h. $C \div 0$ has no meaning.

Exercises 2-8

1. Select the symbols that represent zero:

a. $1 + 0$
b. 0
c. $\frac{4}{0}$
d. $\frac{0}{4}$
e. $5 - 4$
f. $7 - 7$
g. $\frac{a}{0}$
h. $0 + 0$
i. $1 - 1$
j. $100 - 100$
k. $0 \cdot 4$
l. $4 \cdot 0$
m. $0 \cdot 0$
n. $\frac{0}{10}$
o. $\frac{4 - 4}{2}$
p. $\frac{4}{2 - 2}$
q. $\frac{1}{2} - \frac{1}{2}$
r. $14 \cdot 25$
s. $12 \cdot 0$
t. $0 + 12$
u. $2 \cdot (4 + 6 + 0)$
v. $(2 \cdot 4) \div 0$
w. $\frac{4}{4}$
x. $\frac{36}{9} - \frac{36}{12}$

2. Perform the indicated operations, if possible:

a. $376 \cdot 49$

b. $678 \cdot 946$

c. $8984 \div 62$

d. $9484 + 62$

e. $87 \times \$419.98$

f. $69 \times \$876.49$

g. $\$989.26(2 - 2)$

h. $1 \times \$846.25$

i. $5 \times \$14.13$

j. $679 \cdot \frac{4}{4}$

k. $379(146.8 - 145.8)$

l. $(34.6 - 33.6) \times 897$

m. $\$397.16 \div (4 - 3)$

n. $\$897.40 \div (3 - 3)$

o. $(480 \div 24) \div 20$

p. $\$1846 \div (\frac{1}{2} + \frac{1}{2})$

q. $487.97 \times \frac{4}{4} \times 0$

r. $49 \cdot 0 \cdot 47 \cdot 97$

s. $\$97.86 \times 0 \times 0$

t. $(9 - 9) \cdot \frac{7 + 2}{8 + 1}$

u. $976 \cdot 1^6$

v. $1^{12} \times \$97.46$

3. Can you find an error in any of the following statements? a and b are whole numbers.

a. $4 \cdot 0 = 0$

b. $0 \cdot 4 = 0$

c. $2 \cdot 1 = 2$

d. $1 \cdot 2 = 2$

e. If $a \cdot b = 0$, a or $b = 0$

f. If $a \cdot b = 1$, a or $b = 1$

g. If $a \cdot b = 2$, a or $b = 2$

h. If $a \cdot b = 3$, a or $b = 3$

i. If $a \cdot b = C$, a or $b = C$

2-9. Summary.

1. The set of numerals {1, 2, 3, 4, 5, ...} is the set of symbols for the counting numbers.

2. The set of numerals {0, 1, 2, 3, 4, 5, ...} is the set of symbols for the whole numbers.

3. The commutative property for addition: $a + b = b + a$, where a and b represent any whole numbers.

4. The commutative property for multiplication: $a \cdot b = b \cdot a$, where a and b represent any whole numbers.

5. The associative property for addition:
$a + (b + c) = (a + b) + c$ where a, b, c represent any whole numbers.

6. The associative property for multiplication:
$a \cdot (b \cdot c) = (a \cdot b) \cdot c$ where a, b, c represent any whole numbers.

7. The distributive property: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$
where a, b, c are any whole numbers.

8. New symbols: {set of elements}; $>$ is greater than; $<$ is less than; $\neq$ is not equal to.

9. Set and closure. A set is closed under an operation if the combination of any two elements of the set gives an element of the set. The set of counting numbers is closed under addition and multiplication but not under division or subtraction.

10. Inverse operations. Subtraction is the inverse of addition, but subtraction is not always possible in the set of whole numbers. Division is the inverse of multiplication, but division is not always possible in the set of whole numbers; that is, division of one whole number by another whole number does not always yield a whole number.

11. The number line and betweenness. Each whole number is associated with a point on the number line. There is not always a whole number between two whole numbers.

12. Special numbers: 0 and 1 . Zero is the identity for addition; 1 is the identity for multiplication; multiplication by 0 does not have an inverse; division by 0 is not possible.

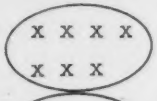
ANSWERS--CHAPTER 1

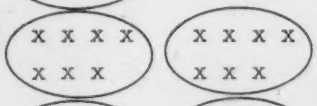
Exercises 1-3.

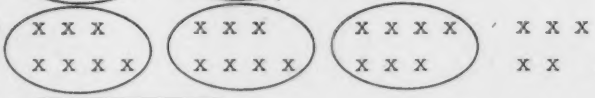
1. (a) 13_{seven}

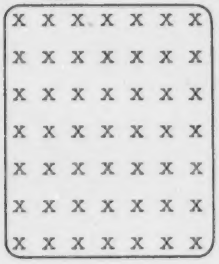
   (b) 24_{seven}

   (c) 116_{seven}

2. (a) 

   (b) 

   (c) 

   (d) 

3. (a) (3 × seven) + (3 × one) = 24

   (b) (4 × seven) + (5 × one) = 33

   (c) (1 × seven × seven) = 49

   (d) (5 × seven × seven) + (2 × seven) + (4 × one) = 263

4. (a) 10_{seven} (d) 163_{seven}

   (b) 11_{seven} (e) 1000_{seven}

   (c) 55_{seven} (f) 1010_{seven}

5. (a) 560_{seven} The 6 means 6 sevens

   (b) 56_{seven} The 6 means 6 ones

   (c) 605_{seven} The 6 means 6(seven × seven)'s or 6(forty-nine)'s

   (d) 6050_{seven} The 6 means 6(seven × seven × seven)'s or 6(three hundred forty-three)'s

6. $seven^4$ or seven to the fourth power

7. The product of 9 sevens or (7^9).

*8. 132_{seven}

9. 452_{seven}

10. 205_{ten}

11. Neither. They are equal.

12. They use seven symbols and seem to have a place value system with base seven. They appear to use | , ∠ , Δ , □ , ⧅ , ⧆ , for 1, 2, 3, 4, 5, 6 and - for zero. ∠∠ follows ∠| .

Exercises 1-4a.

1. (a) 11 (b) 17

3. (a) Yes
   (b) By reading each result from the table and noting results.
   (c) Chart is symmetric with respect to the diagonal.
   (d) 55 different combinations; just a bit over half the total number of combinations.

4.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

5. (a) 28 different combinations. Fewer than 49 because of the commutative law of addition.
   (b) In base seven because there are fewer.
   (c) They are equal since $9 = 12_{seven}$.

Exercises 1-4b.

1. (a) 56_{seven} $(19 + 22 = 41)$
   (b) 110_{seven} $(41 + 15 = 56)$
   (c) 300_{seven} $(109 + 38 = 147)$
   (d) 620_{seven} $(91 + 217 = 308)$

(e) 241_{seven}
(33 + 94 = 127)

(f) 1266_{seven}
(199 + 290 = 489)

(g) 1553_{seven}
(327 + 299 = 626)

(h) 14562_{seven}
(2189 + 1873 = 4062)

(i) 6441_{seven}
(2160 + 123 = 2283)

(j) 1644_{seven}
(327 + 342 = 669)

(k) $14,654_{seven}$
(1917 + 2189 = 4106)

2. (a) 2_{seven} (b) 4_{seven} (c) 4_{seven}

3. (a) 2_{seven}
(7 - 5 = 2)

(b) 36_{seven}
(47 - 20 = 27)

(c) 163_{seven}
(98 - 4 = 94)

(d) 151_{seven}
(91 - 6 = 85)

(e) 6_{seven}
(32 - 26 = 6)

(f) 506_{seven}
(323 - 72 = 251)

(g) 203_{seven}
(247 - 146 = 101)

(h) 406_{seven}
(1715 - 1513 = 202)

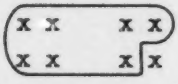
(i) 552_{seven}
(319 - 37 = 282)

(j) 36_{seven}
(74 - 47 = 27)

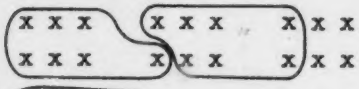
(k) 1254_{seven}
(1261 - 781 = 480)

(l) 54_{seven}
(136 - 97 = 39)

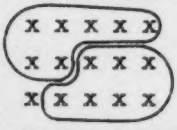
4. (a)



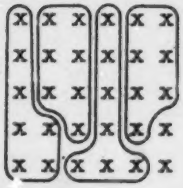
(b)



(c)



(d)



Multiplication, Base Ten

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Exercises 1-4c.

1. Study of this table should emphasize the following:
   (a) The product of 0 and any number is zero.
   (b) The product of 1 and any number is the number itself.
2. The order in multiplication does not affect the product. This is indicated by the fact that the parts of the table on opposite sides of the diagonal line are alike.
3. Multiplication, Base Seven

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

Study of this table is valuable for the additional insight it affords into the understanding of multiplication. There is no value in memorizing it. The table may be used to emphasize that division is the inverse of multiplication.

Exercises 1-4d.

1. (a) 45_{seven} (b) 222_{seven}
   (c) 1116_{seven} (d) 3325_{seven}
   (e) 3464_{seven} (f) $443,115_{seven}$
   (g) $106,533_{seven}$ (h) $5,511,426_{seven}$
   (i) $125,150_{seven}$ (j) $1,660,101_{seven}$

2. (a) 5_{seven} (b) 62_{seven}
   (c) 421_{seven} with a remainder of 2_{seven}.
   (d) 123_{seven} with a remainder of 12_{seven}.

Exercises 1-5.

1. (a) $50_{ten} = (1 \times seven^2) + (0 \times seven) + (1 \times one)$
   $= 101_{seven}$
   (b) $145_{ten} = (2 \times seven^2) + (6 \times seven) + (5 \times one)$
   $= 265_{seven}$
   (c) $1024_{ten} = (2 \times seven^3) + (6 \times seven^2)$
   $+ (6 \times seven) + (2 \times one) = 2662_{seven}$

2. (a) 15_{seven} (d) 104_{seven}
   (b) 51_{seven} (e) 431_{seven}
   (c) 62_{seven} (f) 3564_{seven}

Exercises 1-6.

1. (x x x) (x x x) (x x x) (x x x) (x x x) x

   (a) 5 groups of three and 1 left over.

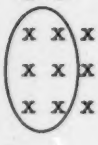
   (b) No. Only the digits "0", "1", and "2" are used in the base three system. "5" is not one of these.

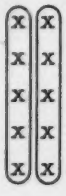
   (c) (1 group of three^2) + (2 groups of three) + (1 left over).

   (d) $16_{ten} = 121_{three}$.

2. (a) , 11_{ten}

   (c) , 11_{ten}

   (b)  , 11_{ten}

   (d)  , 11_{ten}

3.

Base Ten	0	1	2	3	4	5	6	7	8	9	10
Base Five	0	1	2	3	4	10	11	12	13	14	20

Base Ten	11	12	13	14	15	16	17	18	19	20
Base Five	21	22	23	24	30	31	32	33	34	40

Base Ten	21	22	23	24	25	26	27	28	29	30
Base Five	41	42	43	44	100	101	102	103	104	110

4. (a) two (b) two (c) two (d) two

5. (a) $(2 \times 36) + (4 \times 6) + (5 \times 1) = 101$

   (b) $(4 \times 25) + (1 \times 5) + (2 \times 1) = 107$

   (c) $(1 \times 27) + (0 \times 9) + (0 \times 3) + (2 \times 1) = 29$

   (d) $(1 \times 64) + (0 \times 16) + (2 \times 4) + (1 \times 1) = 73$

   Other answers are acceptable, i.e., $(2 \times 6^2) + (4 \times 6^1) + (5 \times 1)$.

6.

	Base Ten	Base Six	Base Five	Base Four	Base Three
(a)	11	15	21	23	102
(b)	15	23	30	33	120
(c)	28	44	103	130	1001
(d)	36	100	121	210	1100

Exercises 1-7.

1.

Base ten	0	1	2	3	4	5	6	7	8	9	10
Base two	0	1	10	11	100	101	110	111	1000	1001	1010

11	12	13	14	15	16	17	18	19	20
1011	1100	1101	1110	1111	10000	10001	10010	10011	10100

21	22	23	24	25	26	27	28	29
10101	10110	10111	11000	11001	11010	11011	11100	11101

30	31	32	33
11110	11111	100000	100001

2. Addition, Base two

+	0	1
0	0	1
1	1	10

There are only four addition "facts."

3. Multiplication, Base Two

×	0	1
0	0	0
1	0	1

There are only four multiplication "facts." The two tables are not alike, except that 0 + 0 and 0 × 0 both equal 0.

The binary system is very simple because there are only four addition and four multiplication "facts" to remember. Computation is simple. Writing large numbers, however, is tedious.

4. (a) $111_{two} = (1 \times two^2) + (1 \times two) + (1 \times one) = 7$

(b) $1000_{two} = (1 \times two^3) + (0 \times two^2) + (0 \times two) + (0 \times one) = (1 \times 2^3) = 8$

(c) $10101_{two} = (1 \times two^4) + (0 \times two^3) + (1 \times two^2) + (0 \times two) + (1 \times one) = (1 \times 2^4) + (1 \times 2^2) + (1 \times 1) = 21$

(d) $11000_{two} = (1 \times two^4) + (1 \times two^3) + (0 \times two^2) + (0 \times two) + (0 \times one) = (1 \times 2^4) + (1 \times 2^3) = 24$

(e) $10100_{two} = (1 \times two^{4}) + (0 \times two^{3}) + (1 \times two^{2}) + (0 \times two) + (0 \times one)$
$= (1 \times 2^{4}) + (1 \times 2^{2}) = 20.$

5. (a) 111_{two} (c) 110000_{two}
(b) 1011_{two} (d) 110110_{two}

6. (a) 10_{two} (c) 111_{two}
(b) 11_{two} (d) 11_{two}

7. (a) 100011_{two} (c) 1100_{two}
(b) 10000000_{two} (d) 1100100_{two}

Answers--Chapter 2

Exercises 2-2a.

1. Parts a, c, d, f, h, i, m are true
Parts b, e, g, j, k, ℓ, are false

2. (a) 644 (b) 110,596 (c) 155,752 (d) 105_{seven}

3. (a) $7 + 4 = 4 + 7$ (f) $(3 \cdot 2) + 5 = 5 + (3 \cdot 2)$
(b) $12 \cdot 5 > 5 \cdot 11$ (g) $8 - 3 < 9 - 3$
(c) $23 \cdot 12 < 12 \cdot 32$ (h) $86 \cdot 135 = 135 \cdot 86$
(d) $3 < 6$ (i) $24 \div 3 > 3 \div 24$
(e) $16 > 9 > 3$ (j) $a > c$

4. (a) 2052 (b) 25,620 (c) 289,884 (d) 1135_{seven}

5. (a) 5 (e) 46
(b) 5 (f) 0, 1, 2, 3, 4, 5, 6
(c) 0 (g) 0, 1, 2, 3, 4
(d) 0, 1 (h) any whole number

Exercises 2-2b.

1. Result is unchanged in parts a, b, c.
Result is changed in parts d, e, f, g, h, i.

2. No.

3. The activities are commutative in parts a, c, e.

4. Yes.

5. The operation in part d is commutative. (The operations in parts a, b, c are commutative only if the first number in each is equal to the second.)

6. Examples of commutative activities.

   To wash your face and wash your hair.
   To go north one block and then west one block.
   To count to 100 and write the alphabet.

   Examples of activities which are not commutative.

   To put out the cat and go to bed.
   To eat dinner and get up from the table.
   To rake the leaves and burn them.

Exercises 2-3.

1. (a) $(4 + 7) + 2 = 4 + (7 + 2)$ Associative property of addition
   $(4 + 7) + 2 = 11 + 2 = 13$
   $4 + (7 + 2) = 4 + 9 = 13$

   (b) $8 + (6 + 3) = (8 + 6) + 3$ Associative property of addition
   $8 + (6 + 3) = 8 + 9 = 17$
   $(8 + 6) + 3 = 14 + 3 = 17$

   (c) $46 + (73 + 98) = (46 + 73) + 98$ Associative property of addition
   $46 + (73 + 98) = 46 + 171 = 217$
   $(46 + 73) + 98 = 119 + 98 = 217$

   (d) $(6 \cdot 5) \cdot 9 = 6 \cdot (5 \cdot 9)$ Associative property of multiplication
   $(6 \cdot 5) \cdot 9 = 30 \cdot 9 = 270$
   $6 \cdot (5 \cdot 9) = 6 \cdot 45 = 270$

   (e) $(21 + 5) + 4 = 21 + (5 + 4)$ Associative property of addition
   $(21 + 5) + 4 = 26 + 4 = 30$
   $21 + (5 + 4) = 21 + 9 = 30$

   (f) $(9 \cdot 7) \cdot 8 = 9 \cdot (7 \cdot 8)$ Associative property of multiplication
   $(9 \cdot 7) \cdot 8 = 63 \cdot 8 = 504$
   $9 \cdot (7 \cdot 8) = 9 \cdot 56 = 504$

(g) $436 + (476 + 1) = (436 + 476) + 1$ — Associative property of addition
$436 + (476 + 1) = 436 + 477 = 913$
$(436 + 476) + 1 = 912 + 1 = 913$

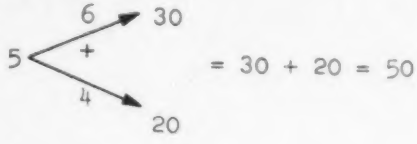
(h) $(57 \cdot 80) \cdot 75 = 57 \cdot (80 \cdot 75)$ — Associative property of multiplication
$(57 \cdot 80) \cdot 75 = (4560) \cdot 75 = 342{,}000$
$57 \cdot (80 \cdot 75) = 57 \cdot 6000 = 342{,}000$

2. (a) No.
(b) No.
(c) There is no associative property of subtraction, or the associative property of subtraction does not hold.

3. (a) No.
(b) No.
(c) $(75 \div 15) \div 5 = 1$
(d) $75 \div (15 \div 5) = 25$
(e) $80 \div (20 \div 2) = 8$
(f) $(80 \div 20) \div 2 = 2$
(g) The associative property does not hold for division.

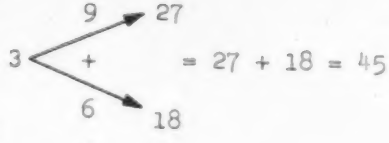
Exercises 2-4.

1. (a)



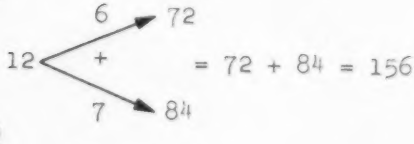
$= 30 + 20 = 50$

(b)



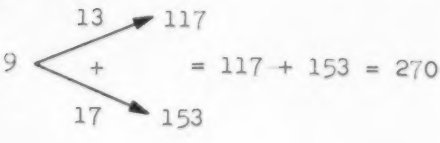
$= 27 + 18 = 45$

(c)



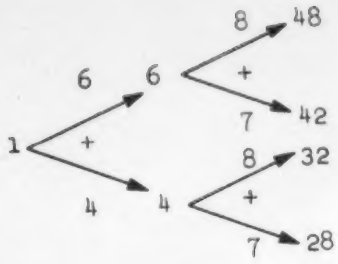
$= 72 + 84 = 156$

(d)



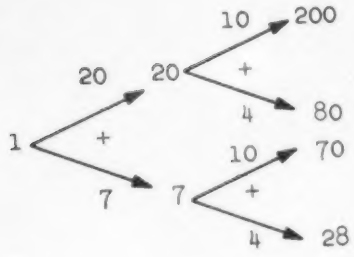
$= 117 + 153 = 270$

(e)



$= (48 + 42) + (32 + 28)$
$= 90 + 60$
$= 150$

(f)



$= (200 + 80) + (70 + 28)$
$= 280 + 98$
$= 378$

2. (a) $4 \cdot 12 = 48$
$28 + 20 = 48$

(b) $18 + 24 = 42$
$6 \cdot 7 = 42$

(c) $48 + 42 = 90$
$15 \cdot 6 = 90$

(d) $23 \cdot 5 = 115$
$46 + 69 = 115$

(e) $11 \cdot 7 = 77$
$33 + 44 = 77$

(f) $30 + 18 = 48$
$6 \cdot 8 = 48$

(g) $2 \cdot 24 = 48$
$32 + 16 = 48$

(h) $12 \cdot 5\frac{1}{4} = 63$
$60 + 3 = 63$

(i) $3216 + 3484 = 6700$
$67 \cdot 100 = 6700$

(j) $36 + 36 = 72$
$72 \cdot 1 = 72$

3. (a) $3 \cdot (4 + \underline{3}) = (3 \cdot 4) + (3 \cdot 3)$

(b) $2 \cdot (\underline{4} + 5) = (2 \cdot 4) + (\underline{2} \cdot 5)$

(c) $13 \cdot (6 + 4) = (13 \cdot \underline{6}) + (13 \cdot \underline{4})$

(d) $(2 \cdot 7) + (3 \cdot \underline{7}) = (\underline{2} + \underline{3}) \cdot 7$

(e) $(\underline{6} \cdot 4) + (\underline{7} \cdot 4) = (6 + 7) \cdot \underline{4}$

4. (a) $9 \cdot (8 + 2)$ (d) $(13 \cdot 6) + (27 \cdot 6)$
   (b) $(8 \cdot 14) + (8 \cdot 17)$ (e) $(15 \cdot 6) + (15 \cdot 13)$
   (c) $(12 \cdot 5) + (12 \cdot 7)$ (f) $12 \cdot (5 + 4)$

5. (a) $(5 \cdot 7) + (5 \cdot 8) = 5 \cdot (7 + 8)$
   (b) $(3 \cdot 4) + (3 \cdot 5) = 3 \cdot (4 + 5)$
   (c) $(5 \cdot 11) + (5 \cdot 2) = 5 \cdot (11 + 2)$
   (d) $(3 \cdot 9) + (3 \cdot 17) = 3 \cdot (9 + 17)$
   (e) $(5 \cdot 20) + (5 \cdot 23) = 5 \cdot (20 + 23)$
   (f) $(3 \cdot 10) + (3 \cdot 7) = 3 \cdot (10 + 7)$

6. The following parts are true: b, c, d.
   The following parts are false: a, e.

Exercises 2-5.

1. The operations in the following parts have inverses:
   a, b, c, d, e, f, h, j, l, o.

2. (a) Put down the pencil.
   (b) Take off your hat.
   (c) Get out of a car.
   (d) Withdraw your hand.
   (e) Divide.
   (f) Tear down.
   (h) Step backward.
   (j) Subtraction.
   (l) Addition.
   (o) Putting on a tire.

3. (a) 46471 (f) $1342.67 (k) 33
   (b) $507.10 (g) 876 (l) 1476
   (c) 506 ft. (h) 987 (m) 68
   (d) $1412.78 (i) 798 (n) 143
   (e) $1101.04 (j) 697 (o) 58140

4. (a) 5
(b) 5
(c) 1
(d) 7
(e) None
(f) 0
(g) 8
(h) 4
(i) 3
(j) 3
(k) None
(l) 7
(m) Any whole number
(n) 4
(o) 5
(p) 5
(q) 9
(r) 9
(s) 6
(t) 0
(u) 0
(v) 0
(w) 0
(x) 1
(y) 1
(z) 1

5. (a) 19
(b) 1992
(c) 89
(d) 19,219
(e) 165821
(f) 13
(g) 6
(h) 20

Exercises 2-6.

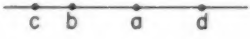
1. (a) 17
(b) 21
(c) 4
(d) 7
(e) None.
(f) 2
(g) None
(h) 88
(i) (3) is answer. The answer could be written as either $a - (b + 1)$ or $(a - b) - 1$ or $(a - 1) - b$.

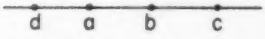
2. (a) 10
(b) 11
(c) 24
(d) 30
(e) 18
(f) 22
(g) 16
(h) 9

3. (a), (b), (c), (g), (i), (j).

4. (a) Yes. (b) Yes. (c) Yes.

5.





Either of the two situations is possible. The diagrams indicate that b is between c and d regardless of whether $c < d$ or $d < c$.

Exercises 2-7.

1. The symbols in the following parts represent the number 1:

   (a), (b), (c), (d), (e), (i), (k), (l), (m), (o), (p).

2. (a) $100 \cdot 1 = 100$ (d) $1 \cdot \frac{2}{3} \cdot 1 = \frac{2}{3}$

   (b) $10 \cdot 1 \cdot 1 \cdot 1 = 10$ (e) $0 \cdot 1 = 0$

   (c) $\frac{14}{1} = 14$ (f) $1 \cdot 0 = 0$

3. We can get any counting number by the repeated addition of 1 to another counting number if the number we wish to get is larger than the counting number to which we add.

   We can get any counting number by the repeated subtraction of 1 from another counting number if the number we wish to get is smaller than the counting number from which we subtract.

4. Yes. $1 - 1 = 0$; $3 - 1 - 1 - 1 = 0$. Zero is not a counting number.

5. The successive addition of 1's to any counting number will give a counting number. But, the successive subtraction of 1's from any counting number will become 0 if carried far enough.

6. (a) 876429 (c) 897638 (e) 3479 (g) 1

   (b) 976538 (d) 896758 (f) 97 (h) 1

Exercises 2-8.

1. The symbols in the following parts represent zero:

   (b), (d), (f), (h), (i), (j), (k), (l), (m), (n), (o), (q), (s).

2.

(a)	18424	(l)	897
(b)	641388	(m)	$397.16
(c)	144, remainder 56	(n)	Division by zero not possible
(d)	152, remainder 60	(o)	1
(e)	$36538.26	(p)	$1846
(f)	$60477.81	(q)	0
(g)	0	(r)	0
(h)	$846.25	(s)	0
(i)	$70.65	(t)	0
(j)	679	(u)	976
(k)	379	(v)	$97.46

3. The error is in the generalization to C in part (i). If $a \cdot b = C$, a or b does not need to be C. Example: $2 \cdot 2 = 4$. This exercise shows the error that may be made by making a generalization on a few cases.